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# A model for emergence of space and time

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ABSTRACT

We study string field theory (third quantization) of the two-dimensional model of quantum geometry called generalized CDT ("causal dynamical triangulations"). Like in standard non-critical string theory the so-called string field Hamiltonian of generalized CDT can be associated with $W$-algebra generators through the string mode expansion. This allows us to define an "absolute" vacuum. "Physical" vacua appear as coherent states created by vertex operators acting on the absolute vacuum. Each coherent state corresponds to specific values of the coupling constants of generalized CDT. The cosmological "time" only exists relatively to a given "physical" vacuum and comes into existence before space, which is created because the "physical" vacuum is unstable. Thus each CDT "universe" is created as a "Big Bang" from the absolute vacuum, its time evolution is governed by the CDT string field Hamiltonian with given coupling constants, and one can imagine interactions between CDT universes with different coupling constants ("fourth quantization")



## 1. Introduction

Two-dimensional models can be useful when it comes to addressing a number of conceptual issues related to the quantization of geometry, simply because the corresponding quantum field theory is well defined and explicit calculations can be performed. Here we will consider the model of quantum geometry denoted "causal dynamical triangulations" (CDT) [1]. The name refers to the regularization of the continuum theory, which is regularized by triangulating spacetime in a specific way using the path integral formalism. The continuum limit is obtained when the cut-off, the link length $a$ used in the triangulations, is removed. This limit is well defined and corresponds to quantized two-dimensional Hořava–Lifshitz gravity[1] (for Hořava–Lifshitz gravity see [4], for the CDT connection see [5]).

In two-dimensional CDT it is assumed that space, which is one-dimensional, has the topology of a circle. CDT then describes the quantum "propagation" of space as a function of time. Here we will consider so-called generalized CDT (GCDT) where one allows space to split and join into disconnected circles as a function of time [6]. A complete "string field theory" which allows us to calculate any such amplitude has been developed [7]. It is inspired by the string field theory for non-critical string theory [8–10]. Both theories are perturbative theories in the topology of the spacetime connecting the "incoming" ("initial") spatial boundaries and the "outgoing" ("final") spatial boundaries.[2]

In the case of non-critical string field theory $W$-algebras play an important role and are intimately related to integrable KP hierarchies associated with non-critical string theories [13,10]. In the case of GCDT this relation is not yet fully developed, but most likely it exists. Multicritical GCDT and Ising models coupled to GCDT can be formulated [14] and the associated $W$-algebras can be identified [15]. However, here we will concentrate on the very simplest GCDT model, its associated $W$-algebra and a possible physical interpretation. In Section 2 we show how the $W$-algebra appears in GCDT and we discuss how the simplest $W$-Hamiltonian, being a Hamiltonian with no coupling constants and no spacetime interpretation, contains the string field theory of GCDT and the seeds for a Big Bang. Section 3 contains conclusion and discussion.

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[1] CDT can be formulated also in higher dimensions, and also in that case there is seemingly a continuum limit of the regulated theory (see [2] for the original articles, [3] for a recent review).

[2] It is even possible to perform certain sums over all topologies, both in the case of non-critical string field theories [11] and in the case of GCDT [12].

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## 2. The $W$- and GCDT Hamiltonians

The formal definition of a $W^{(3)}$ algebra in terms of operators $\alpha_n$ satisfying

$$[\alpha_m, \alpha_n] = m\,\delta_{0,n+m} \tag{1}$$

is the following

$$\alpha(z) = \sum_{n\in Z} \frac{\alpha_n}{z^{n+1}}, \quad W^{(3)}(z) = \frac{1}{3} :\alpha(z)^3: = \sum_{n\in Z} \frac{W_n^{(3)}}{z^{n+3}}. \tag{2}$$

The normal ordering :(·): refers to the $\alpha_n$ operators ($\alpha_n$ to the left of $\alpha_m$ for $n > m$)[3] and we have

$$W_n^{(3)} = \frac{1}{3} \sum_{a+b+c=n} :\alpha_a\alpha_b\alpha_c: . \tag{3}$$

In the $W^{(3)}$-algebra related to non-critical string field theory $\alpha_0$ is identical zero (see [10] for details), but in the GCDT case $\alpha_0$ plays a special role and we thus write

$$\alpha_n = \begin{cases} a_n^\dagger & [n>0] \\ p & [n=0] \\ -na_{-n} & [n<0] \end{cases} \tag{4}$$

where the operators satisfy

$$[\,a_m\,,\, a_n^\dagger\,] = \delta_{m,n} \qquad [\,a_m,\, a_n\,] = [\,a_m^\dagger,\, a_n^\dagger\,] = 0 \tag{5}$$

$$[\,q\,,\, p\,] = i, \qquad [\,q\,,\, q\,] = [\,p\,,\, p\,] = 0 \tag{6}$$

$$[\,p\,,\, a_n^\dagger\,] = [\,p\,,\, a_n\,] = [\,q\,,\, a_n^\dagger\,] = [\,q\,,\, a_n\,] = 0. \tag{7}$$

In (6) and (7) we have introduced an operator $q$ conjugate to $p = \alpha_0$. We then define the "absolute vacuum" $|0\rangle$ by the following condition:

$$a_n|0\rangle = p|0\rangle = 0 \qquad [n = 1, 2, \ldots], \tag{8}$$

and the so-called $W$-Hamiltonian $\hat{H}_W$:

$$\begin{aligned} \hat{H}_{\mathrm{W}} &:= -W_{-2}^{(3)} \\ &= -\sum_{\substack{n,m,l \\ [n+m+2=l]}} a_n^\dagger a_m^\dagger l a_l - \sum_{\substack{n,m,l \\ [n+2=m+l]}} a_n^\dagger m a_m l a_l \\ &\quad - 2\sum_{\substack{n,l \\ [n+2=l]}} p a_n^\dagger l a_l - p a_1 a_1 - 2p^2 a_2 . \end{aligned} \tag{9}$$

Note that $\hat{H}_W$ does not contain any coupling constants.

Related to $\hat{H}_W$ and the absolute vacuum we now define a generating functional with sources $x$, $y$

$$Z[x,y;T] := \langle 0|\exp\Big(\sum_{n=1}^{\infty} y_n a_n\Big) e^{-T\hat{H}_{\mathrm{W}}} \exp\Big(\sum_{n=1}^{\infty} x_n a_n^\dagger\Big)|0\rangle \tag{10}$$

The states in the Hilbert space $\mathcal{H}$ associated with $\hat{H}_W$ are obtained by acting repeatedly on the absolute vacuum $|0\rangle$ with the operators $a_n^\dagger$ and $q$. Such an "initial" state is then "propagated" a "time" $T$ and projected onto a similar "final" state. These amplitudes can be obtained from the generating functional $Z[x,y;T]$ by differentiation with respect to $x$ and $y$. However, we should stress that at this point there is no compelling reason to denote $T$ a (Euclidean) time and the form of $\hat{H}_W$ does not suggest any obvious geometry-interpretation. One could equally well view $T$ as an "inverse temperature" and use $Z[x,y;T]$ to calculate the corresponding partition function. Here we will view the states and dynamics associated with $\hat{H}_W$ as "pre-geometry", and only by a projection onto a subspace of $\mathcal{H}$ the parameter $T$ will get an interpretation as (Euclidean) time and the states will obtain an interpretation as spatial geometries, and the amplitudes will then be probability amplitudes for the propagation of spatial geometries in (Euclidean) time. This reinterpretation of $\hat{H}_W$ will be made by relating it to the standard string field Hamiltonian $\hat{H}$ of GCDT defined relatively to a "physical" vacuum $|\mathrm{vac}\rangle$.

Recall the following representation of the GCDT $\hat{H}$ (the one originally used in [7]):

$$\begin{aligned} \hat{H} &= \hat{H}_0 - g\int dL_1 \int dL_2\, \Psi^\dagger(L_1)\Psi^\dagger(L_2)\,(L_1+L_2)\Psi(L_1+L_2) \\ &\quad - gG\int dL_1 \int dL_2\, \Psi^\dagger(L_1+L_2)\, L_2\Psi(L_2)\, L_1\Psi(L_1) \\ &\quad - \int dL \rho(L)\Psi(L), \end{aligned} \tag{11}$$

where

$$\hat{H}_0 = \int_0^\infty dL\, \Psi^\dagger(L) H_0 \Psi(L), \qquad H_0 = -\frac{\partial^2}{\partial L^2} L + \mu L,$$

$$\rho(L) = \delta(L), \tag{12}$$

and where the operators $\Psi(L)$ and $\Psi^\dagger(L)$ satisfy

$$[\Psi(L), \Psi^\dagger(L')] = \delta(L-L'), \qquad \Psi(L)|\mathrm{vac}\rangle = 0. \tag{13}$$

In (11) $\Psi^\dagger(L)$ creates a spatial universe of length $L$ from the physical vacuum $|\mathrm{vac}\rangle$. The vectors $|L\rangle = \Psi^\dagger(L)|\mathrm{vac}\rangle$, $L$ positive, span the Hilbert space where $H_0$ is defined (see [7] for details). $\hat{H}$ represents a third quantization in the sense that space can be created from the vacuum $|\mathrm{vac}\rangle$ by acting with $\Psi^\dagger(L)$ and annihilated by acting with $\Psi(L)$. Thus $\hat{H}_0$ propagates spatial slices in time, can change their lengths but cannot merge or split the spatial splices. $\mu$ denotes the cosmological constant and acts to limit the growth of the spatial universe. The second term on the rhs of (11) splits a spatial slice of length $L_1 + L_2$ in two slices of lengths $L_1$ and $L_2$, governed by a coupling constant $g$ of mass dimension 3. The third term on the rhs of (11) merges two spatial slices of length $L_1$ and $L_2$ into one slice of length $L_1 + L_2$, governed by a coupling constant $g \cdot G$, where $G$ is dimensionless and is introduced to allow for a potential asymmetry between splitting and joining. Finally the fourth term on the rhs of (11) is a tadpole term which allows a spatial slice to disappear into the vacuum, but only if its length is zero. Thus the interaction terms in $\hat{H}$ preserve the total length of the spatial slices and any expansion or contraction of the universe is caused by $\hat{H}_0$ and the coupling constant for topology change of spacetime is $g^2 G$. $\hat{H}$ is not Hermitian because of the tadpole term (and also if $G \neq 1$), but that is always the case

[3] We remark that this ordering is opposite to the standard ordering one would use in conformal field theory. One can obtain the conventional ordering by the so-called ⋆-operation [10], where one uses generating functionals like (10) to express the action of the $\alpha_n$ operators by the action of differential operators acting on the sources of the generating functional. That is also the more precise way the $W^{(3)}$ algebra becomes related to integrable KP-hierarchies in non-critical string theory: certain Dyson–Schwinger equations satisfied by the generating functionals combined with the $W^{(3)}$-algebra properties of the $\alpha_n$'s represented as differential operators acting on the sources ensure integrability (see i.e. [10] for details for non-critical string field theory and [15] for GCDT string field theory). Here we omit for transparency these technical details.

for non-critical string field theory and is enforced upon us by the requirement of stability of the vacuum.

We now make a so-called *mode expansion* of $\hat{H}$. The modes $\phi_n$, $\phi_n^\dagger$ are defined as follows

$$\Psi(-\zeta) = \sum_{n=1}^{\infty} \phi_n \zeta^n, \quad \Psi^\dagger(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{\phi_n^\dagger}{\zeta^{n+1}} \tag{14}$$

where

$$\Psi^\dagger(\zeta) = \int_0^\infty dL\, e^{-\zeta L} \Psi^\dagger(L), \tag{15}$$

and similar for $\Psi$. By construction we have

$$\phi_n|\mathrm{vac}\rangle = 0, \quad [\phi_n, \phi_m^\dagger] = \delta_{n,m}. \tag{16}$$

and after some algebra (see [15] for more details and mode expansions also for GCDT coupled to matter) we obtain

$$\begin{aligned}\hat{H} &= \mu\phi_1 - 2g\phi_2 - gG\phi_1\phi_1 - \sum_{l=1}^{\infty} \phi_{l+1}^\dagger l\phi_l + \mu \sum_{l=2}^{\infty} \phi_{l-1}^\dagger l\phi_l \\ &\quad - 2g \sum_{l=3}^{\infty} \phi_{l-2}^\dagger l\phi_l - g \sum_{l=4}^{\infty} \sum_{n=1}^{l-3} \phi_n^\dagger \phi_{l-n-2}^\dagger l\phi_l \\ &\quad - gG \sum_{l=1}^{\infty} \sum_{m=\max(3-l,1)}^{\infty} \phi_{m+l-2}^\dagger m\phi_m l\phi_l.\end{aligned} \tag{17}$$

Let us now relate the physical vacuum $|\mathrm{vac}\rangle$ to the absolute vacuum $|0\rangle$ and $\hat{H}_W$ to $\hat{H}$. We define the physical vacuum as the following coherent state relative to the absolute vacuum:

$$|\nu\rangle = \mathrm{e}^{i\nu q}|0\rangle, \quad |\mathrm{vac}\rangle_\nu = V(\lambda_1, \lambda_3)\,|\nu\rangle, \tag{18}$$

$$V(\lambda_1, \lambda_3) := \exp\left(-\frac{|\lambda_1|^2}{2} - \frac{|\lambda_3|^2}{2} + \lambda_1 a_1^\dagger + \lambda_3 a_3^\dagger\right) \tag{19}$$

and we have

$$\begin{aligned}&a_1|\mathrm{vac}\rangle_\nu = \lambda_1|\mathrm{vac}\rangle_\nu \qquad a_3|\mathrm{vac}\rangle_\nu = \lambda_3|\mathrm{vac}\rangle_\nu \\ &p|\mathrm{vac}\rangle_\nu = \nu|\mathrm{vac}\rangle_\nu.\end{aligned} \tag{20}$$

From eq. (20) it follows that if we choose

$$\lambda_1 = -\frac{\mu}{2g\sqrt{G}} \qquad \lambda_3 = \frac{1}{6g\sqrt{G}} \qquad \nu = \frac{1}{\sqrt{G}} \tag{21}$$

and make the identification

$$\begin{aligned}a_n \to V(\lambda_1, \lambda_3)\, a_n V^{-1}(\lambda_1, \lambda_3) &= a_n - \lambda_1\delta_{n,1} - \lambda_3\delta_{n,3} \\ &:= \sqrt{G}\,\phi_n\end{aligned} \tag{22}$$

$$a_n^\dagger \to V(\lambda_1, \lambda_3)\, a_n^\dagger V^{-1}(\lambda_1, \lambda_3) := \frac{1}{\sqrt{G}}\,\phi_n^\dagger \tag{23}$$

then eqs. (5) and (8) become consistent with (16). We can finally write

$$g\sqrt{G}\,\hat{H}_{\mathrm{W}}\Big|_{p=1/\sqrt{G}} = \hat{H} - \frac{1}{G}\left(\frac{\mu^2}{4g} + \frac{1}{4g}\phi_4^\dagger - \frac{\mu}{2g}\phi_2^\dagger + \phi_1^\dagger\right) \tag{24}$$

valid on the subspace of $\mathcal{H}$ where the eigenvalue of $p$ is $1/\sqrt{G}$. This is our basic relation. By acting with the vertex operator $V(\lambda_1, \lambda_3)$ defined in (19) on the absolute vacuum $|0\rangle$ we create a condensation of $\phi_1^\dagger$, $\phi_3^\dagger$ and $q$ modes. This condensate defines the coupling constants of a GCDT string field theory, but if our starting point is $\hat{H}_W$ the corresponding *GCDT* vacuum $|\mathrm{vac}\rangle_\nu$ is unstable, as is clear from (24).

It is the condensation of $\phi_3^\dagger$ which creates a non-zero $\lambda_3$ and it is this non-zero $\lambda_3$ which results in the appearance of the term $-\sum_{l=1}^{\infty} \phi_{l+1}^\dagger l\phi_l$. Such a term is necessary if we want to have the possibility of an expanding universe. In the physical vacuum $|\mathrm{vac}\rangle$ the universe can thus both expand and contract and the parameter $T$ multiplying the Hamiltonian can then be interpreted as the time-evolution parameter of the universe. One can say that time $T$ refers to a vacuum $|\mathrm{vac}\rangle_\nu$ and only allows for an interpretation as the cosmological time of a spacetime after $|\mathrm{vac}\rangle_\nu$ is introduced.

## 3. Discussion

We have attempted to create a model of the universe where there is an "absolute" vacuum $|0\rangle$ and a "pre-geometry" Hamiltonian $\hat{H}_W$. We were inspired by non-critical string field theory to choose the simplest possible non-trivial $\hat{H}_W$, related to the $W^{(3)}$ algebra. The corresponding partition function (10) can most likely be related to a tau-functions of a KP hierarchy (details are being worked out), but as mentioned the system does not offer an obvious interpretation as a dynamical system for spacetime. However, acting with a vertex operator on the absolute vacuum brings us to a coherent state (18), $|\mathrm{vac}\rangle_\nu$, which has non-zero overlap to the absolute vacuum. We denote $|\mathrm{vac}\rangle_\nu$ a "physical" vacuum because the corresponding action (22) on creation and annihilation operators, which amounts to a simple shift of expectation values of the operators in $\hat{H}_W$, leads to an interpretation of $\hat{H}_W$ as a Hamiltonian which creates, annihilates and changes space, thus creating a dynamical spacetime, relative to this physical vacuum. At the same time the simple shifts of expectation values define the coupling constants of the string field Hamiltonian which governs the evolution. Clearly this process has some resemblance to standard spontaneous symmetry breaking where the vacuum expectation values of a field might define the values of some of the coupling constants of the theory. At the same time this "symmetry breaking" becomes the source of a "Big Bang", the creation of a universe from nothing since $\hat{H}_W|_{p=1/\sqrt{G}}$ contains the creation operators which will act non-trivially on $|\mathrm{vac}\rangle_\nu$. Once the choice of $|\mathrm{vac}\rangle_\nu$ is made $T$ can be viewed as a cosmological time and space can next be created due to the instability of $|\mathrm{vac}\rangle_\nu$ with respect to $\hat{H}_W$. The origins of space and time are thus different in our model, time being a "precursor" for space, a point also emphasized in [16] although from a different perspective. Many universes can be created and they can join and split as a function of $T$ and we can explicitly calculate such amplitudes [7]. Let $\mathcal{H}(\lambda_1, \lambda_3, \nu)$ be the Fock space spanned by states obtained by acting repeatedly with the $\phi_n^\dagger$ operators on $|\mathrm{vac}\rangle_\nu$. In the larger Hilbert space $\mathcal{H}$ of $\hat{H}_W$ we have that

$$\mathcal{H}(\lambda_1', \lambda_3', \nu') \perp \mathcal{H}(\lambda_1, \lambda_3, \nu) \quad \text{for } \nu' \neq \nu \tag{25}$$

since the operator $p$ is Hermitian. However, all Hilbert spaces with the same value of $\nu$ but different values of $\lambda_1$ and $\lambda_3$ are identical since the overlaps between different coherent states created by acting with $V(\lambda_1, \lambda_3)$ for different values of $\lambda_1$ and $\lambda_3$ are non-zero. Thus universes with different coupling constants can in principle interact if we can provide a suitable interaction term and this interaction could change the values of the coupling constants of the universes. One could call such a scenario a "fourth quantization" since our string field theory is already a "third quantization" as mentioned above. One could imagine to use such change in coupling constants to explain aspects of inflation, provided suitable higer-dimensional models can be consistently formulated [15].

This brings us to a missing ingredient in our construction, namely a mechanism for choosing a specific physical vacuum $|\mathrm{vac}\rangle_\nu$. Being minimalistic one could say that the probability $P(\lambda_1, \lambda_2)$ of being in a universe corresponding to a given choice of cosmological constant and a given choice of coupling constant $g$ would be given related to the overlap between $|0\rangle$ and $|\mathrm{vac}\rangle_\nu$, i.e.

$$P(\lambda_1, \lambda_3) \propto \Big|\langle 0|\mathrm{vac}\rangle_\nu\Big|^2 \propto \mathrm{e}^{-\lambda_1^2-\lambda_3^2} \qquad (26)$$

where the relation between coupling constants and $\lambda_1$ and $\lambda_3$ is given by eqs. (21), but it would be desirable to have a dynamical mechanism for selecting $|\mathrm{vac}\rangle_\nu$. Also, a statement like (26) does not make much sense if one allows interactions between universes with different coupling constants.

It would be interesting to generalize the model to include matter, in particular in such a way that the choice of physical vacuum $|\mathrm{vac}\rangle$ would not only be a choice of the coupling constants related to geometry but also a choice of matter content. Understanding the mechanism for the choice of such $|\mathrm{vac}\rangle$ would be exciting. Equally exciting is the possibility to extend the considerations here to genuine four-dimensional models. All this indeed seems possible [15].

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