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A B S T R A C T

In the $\text{AdS}_5/\text{CFT}_4$ set-up, extremal three-point functions involving two giant 1/2 BPS gravitons and one point-like 1/2 BPS graviton, when calculated using semi-classical string theory methods, match the corresponding three-point functions obtained in the tree-level gauge theory. The string theory computation relies on a certain regularization procedure whose justification is based on the match between gauge and string theory. We revisit the regularization procedure and reformulate it in a way which allows a generalization to the ABJM set-up where three-point functions of 1/2 BPS operators are not protected and where a match between tree-level gauge theory and semi-classical string theory is hence not expected.

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After the successful application of integrability techniques to the planar spectral problem of the $\text{AdS}_5/\text{CFT}_4$ set-up [1], the calculation of three-point functions in the same set-up has attracted renewed attention with some recent highlights being the conjecture of an all loop formula for three-point functions of single trace operators in certain subsectors of $\mathcal{N} = 4$ SYM [2] and the formulation of certain integrability axioms for the cubic string theory vertex [3].

We will be considering three-point functions which do not belong to the class of three-point functions considered in the above references. Our three-point functions involve giant gravitons which in the string theory language correspond to higher dimensional D-or M-branes wrapping certain submanifolds of the string theory background and which in the gauge theory picture are represented by specific linear combinations of multi-trace operators, namely Schur polynomials. Remaining in the gauge theory picture, our three-point functions will involve two Schur polynomials and one single trace operator, all of 1/2 BPS type. Furthermore, the three operators will be chosen such that $\Delta_1 = \Delta_2 + \Delta_3$, where the $\Delta$'s are the conformal dimensions of the operators. Such three-point functions are denoted as extremal three-point functions and are known to require special care in the comparison between gauge and string theory [4]. On the gauge theory side the three-point functions of interest can be calculated using techniques from zero-dimensional field theories [5] (see also [6]) and on the string theory side they can be determined by generalizing a method developed for the calculation of heavy-heavy-light correlators [7–9] from string states to membranes [5].

In the case of the $\text{AdS}_5 \times S^5$ correspondence the 1/2-BPS nature of the operators involved implies that the three-point function considered is protected and thus should take the same value whether calculated in string theory or in gauge theory. As pointed out in [10] the need for special treatment of extremal correlators in string theory is relevant here and in [11] a regularization procedure for the string theory computation which led to the desired match between gauge and string theory was presented.

The $\text{AdS}_4 \times \mathbb{C}P^3$ set-up [12] allows one to consider a similar correlator i.e. an extremal three-point function involving two 1/2 BPS giant gravitons in combination with one 1/2 BPS point-like graviton and the methods developed in [5] for the $\text{AdS}_5/\text{CFT}_4$ calculation can be generalized to this case as well [13]. One remaining subtle point is the choice of regularization procedure in the string theory computation. In the $\text{AdS}_4 \times \mathbb{C}P^3$ correspondence three-point functions of 1/2 BPS operators are not protected and hence in this set-up we cannot expect a match between gauge and string theory results. In particular, this means that on one hand we cannot justify our choice of regulator by a match between the gauge and string theory results but on the other hand a computation of the correlator in the weakly coupled string theory will provide us with a non-trivial prediction about the behaviour of the correlator in the dual strongly coupled field theory. Below we will revisit the regularization procedure employed for the $\text{AdS}_5 \times S^5$
computation and modify it in a way that allows us to generalize it to the $AdS_4 \times \mathbb{CP}^3$ case. Subsequently, we carry out the string theory calculation of the extremal three-point function involving two giant 1/2 BPS gravitons and one point-like 1/2 BPS graviton in $AdS_4 \times \mathbb{CP}^3$.

1. Giant graviton correlators in $AdS_5 \times S^5$ revisited

Giant gravitons in $AdS_5 \times S^5$ are D3-branes which wrap an $S^3$ which constitutes a subset of either $AdS_5$ or $S^5$ [14–16]. We will consider the giants for which the wrapped sphere $S^3 \subset S^5$ and which spin along a circle in the $S^2$ while being located at the center of $AdS_5$. For such giants the dual gauge theory operators are Schur polynomials built on completely anti-symmetric Young diagrams containing a complex scalar field that we will denote as $Z$ [17,6]. The D3-brane action is (in units where the $AdS$ radius has been set to 1)

$$S_{D3} = -\frac{N}{2\pi^2} \int d^4 \sigma \left( \sqrt{-g} - P[C_4] \right),$$

where $g_{\mu\nu} = \delta_{MN} g_{\mu\nu}$, with $a, b = 0, \ldots, 3$ being the world-volume coordinates and $X^M$ the embedding coordinates. Working in global $AdS$ coordinates

$$ds^2 = -\cosh^2 \theta \, dt^2 + d\rho^2 + \sinh^2 \theta \, d\Omega^2_3$$

$$+ d\theta^2 + \sin^2 \theta \, d\theta d\phi + \cos^2 \theta \, d\Omega^2_3,$$

the four-form potential $C_4$ can be written as [15]

$$C_4 = \cos^4 \theta \, \text{Vol}(\Omega_3),$$

where $X_i$ are the angles of the wrapped sphere, i.e. $d\Omega^2_3 = dX_i^2 + \sin^2 \chi_i d\chi_i^2 + \cos^2 \chi_i d\chi_i^2$. Using the ansatz

$$\rho = 0, \quad \sigma^0 = t, \quad \phi = \phi(t), \quad \sigma^1 = \chi_i,$$

one can show that a D3-brane with angular momentum $k$ is stable when it sits at

$$\cos^2 \theta = k/N,$$

and spins at the speed of light, $\phi = 1$ [14]. In order to determine the three-point function of two giant gravitons and a point-like graviton (i.e. a chiral primary) we should determine the variation of the Euclidean version of the D3-brane action in response to the insertion of the desired chiral primary at the boundary of $AdS_5$ and subsequently evaluate these fluctuations on the Wick rotated giant graviton solution. As this procedure was described in detail in [5] we shall be brief here. Denoting the spherical harmonic representing the point-like graviton as $Y_\Delta$ (with $\Delta$ referring to its conformal dimension) we can write the variation of the D3-brane action as [5]

$$\delta S = \frac{N}{2\pi^2} \cos^2 \theta \int d^4 \sigma \left( \frac{2}{\Delta + 1} Y_\Delta (\partial_\Delta^2 - \Delta^2) s^\Lambda \right. + 4 \left[ \Delta \cos^2 \theta - \sin \theta \cos \theta \partial_\theta \right] s^\Lambda Y_\Delta),$$

where $s^\Lambda$ is the bulk to boundary propagator. As our spherical harmonic we choose

$$Y_\Delta = \frac{\sin^\Delta \theta e^{i\Delta \phi}}{2\Delta^{3/2}},$$

which corresponds to the single trace operator $Tr Z^\Delta$ in the gauge theory language. With this choice for $Y_\Delta$ the first term in eq. (6) is finite and gives the following contribution to the three-point function structure constant

$$C^3_{\text{finite}} = -\frac{k}{N} \left( 1 - \frac{k}{N} \right)^{\Delta/2},$$

whereas the contribution coming from the term with square brackets takes the form of a divergent integral with a pre-factor which tends to zero. In Ref. [11] it was proposed to regularize the divergent integral by replacing the simple spherical harmonic $Y_\Delta$ with the more involved one

$$Y_{\Delta+l; \Delta} = N^\Delta_{\Delta+l} \sin^\Delta \theta e^{i\Delta \phi} F_2 (-l, \Delta + l + 2; \Delta + 1; \sin^2 \theta),$$

where $N^\Delta_{\Delta+l}$ is a normalization factor, and to consider the limit $l \to 0$ where $Y_{\Delta+l; \Delta} \to Y_\Delta$, and where the contribution from the ill defined term of eq. (6) becomes finite. This procedure leads to a match between the string and gauge theory computation. The obtained match justifies the choice of the regularization procedure but does not suggest a general principle that one could build on when aiming at a generalization to giant gravitons in $AdS_4 \times \mathbb{CP}^3$.

One property which characterizes the spherical harmonic (9) is that it extends the simple one without making use of additional coordinates on $S^5$. However, this property is somewhat deceptive and is not the correct clue to an extension to the $AdS_4 \times \mathbb{CP}^3$ setup.

Here we shall formulate the regularization procedure in a slightly different manner which will allow us to generalize it to the latter set-up. For that purpose we make use of the fact that spherical harmonics on $S^5$ are in one-to-one correspondence with symmetric traceless SO(6) tensors. In particular (leaving out normalization factors) the spherical harmonic (7), which translates into $Tr Z^\Lambda$ in the field theory, corresponds to the tensor

$$C_{\Delta-k} \ldots (\Delta-k) = i^k,$$

where symmetrization is understood. It is easy to show that adding more indices of type 1 and type 2 to the tensor (i.e. adding more fields of type $\Phi_1$ and $\Phi_2$ to the operator) does not regularize the divergent integral. However, one can regularize the integral by considering the following symmetric traceless tensor

$$C_{\Delta-k} \ldots (\Delta-k) = i^{k+n},$$

where $n < 2l$ and subsequently taking the limit $l \to 0$. Obviously, the gauge theory operator resulting from this tensor involves the complex field $\chi = \Phi_1 + i\Phi_2$, in addition to the complex field $Z$. It is easy to check that the spherical harmonic (9) corresponds to an operator involving all six scalar fields of $N = 4$ SYM but it is not straightforward to express the corresponding C-tensor in a closed form. The tensor (11) translates into the following spherical harmonic

$$Y_{\Delta+l; \Delta} = N^\Delta_{\Delta+l} \sin^\Delta \theta e^{i\Delta \phi} (l \theta) \sin^2 (l \chi_1) e^{2i l \chi_2},$$

where $N^\Delta_{\Delta+l}$ is another normalization constant. Using this spherical harmonic instead of $Y_\Delta$ when evaluating the second line of (6) and subsequently taking the limit $l \to 0$ gives us the following result for the regularized contribution to the three-point function

$^{1}$ The chiral primary operators of $N = 4$ SYM can be written in the form $C_{\Delta-k}^{(1/2)}(\Phi_1 + i\Phi_2)$ where the $\Phi_5$ can be any of the six real scalar fields and where $C_{\chi}$ is a symmetric traceless tensor. We take the complex scalar field $Z$ to be given by $Z = \Phi_1 + i\Phi_2$. 

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The coordinates theory a\follows the three-point function involves two giant gravitons and a point-like graviton (a chiral primary) we should again evaluate the variation of the Euclidean action in response to the insertion of the chiral primary at the boundary of AdS. Denoting as before the spherical harmonic representing the chiral primary as \( Y_\Delta \), we can write the variation of the action as

\[
\delta S_{DBI} = c \cdot \int d^6\sigma e^{ip} \sqrt{g} \left[ \epsilon \left( 2\Delta + \frac{\cosh 2\rho}{\cosh 2\rho - 2\alpha^2 \omega^2} \right) \left( \frac{4}{\Delta + 2} \right)^2 \right] \left[ Y_\Delta(\Omega) \right] s^\Delta(X),
\]

where \( c = \frac{8}{\sqrt{\Delta}} \). The remaining part of the contribution from eqs. (24) and (25) is ill-defined. In the same manner as it was the case for the AdS\(5 \times S^5 \) set-up the ill-defined part takes the form of a divergent integral times a vanishing pre-factor. This again calls for a regularization of the contribution. With an eye to the regularization of Ref. [11] one would be tempted to look for a regulator in the form of a solution of the Laplace equation which would reduce to the simple one (26) when some of its quantum numbers were sent to zero and which would involve as few as possible additional coordinates compared to (26). However, the Laplace equation when expressed in the coordinates (15)–(16) can only in a few special cases be solved by separation of variables and the corresponding spherical harmonics turn out not to regularize the three-point function.

\[
\delta S_{WZ} = -c \cdot \int d^6\sigma \omega \sqrt{g} s^\Delta(X),
\]

As our spherical harmonic we would like to use

\[
Y_\Delta = \left( r^2 e^{i\chi} \right)^{\Delta/2},
\]

which corresponds to the single trace operator \( \text{Tr}(Z_1 Z_2) \) containing the same fields as the giant graviton. With this choice of spherical harmonic the contribution to the three-point function from the term containing square brackets is finite and yields

\[
C_{\text{finite}} = \frac{1}{N} \left( \frac{\lambda}{2\pi r} \right)^{1/4} (2k)(2\alpha^2)^{\Delta/2} \Delta + 1,
\]

where \( \lambda = N/m \). The remaining part of the contribution from eqs. (24) and (25) is ill-defined. In the same manner as it was the case for the AdS\(5 \times S^5 \) set-up the ill-defined part takes the form of a divergent integral times a vanishing pre-factor. This again calls for a regularization of the contribution. With an eye to the regularization of Ref. [11] one would be tempted to look for a regulator in the form of a solution of the Laplace equation which would reduce to the simple one (26) when some of its quantum numbers were sent to zero and which would involve as few as possible additional coordinates compared to (26). However, the Laplace equation when expressed in the coordinates (15)–(16) can only in a few special cases be solved by separation of variables and the corresponding spherical harmonics turn out not to regularize the three-point function.

\[
C_{\text{regularized}} = \frac{1}{\sqrt{\Delta}} \left( 1 + \frac{k}{N} \right)^{\Delta/2},
\]

meaning that the total three-point function takes the form

\[
C_{\text{total}} = \frac{1}{\sqrt{\Delta}} \left( 1 - \frac{k}{N} \right)^{\Delta/2},
\]

which precisely coincides with the gauge theory result obtained in [5]. One can also extend the C-tensor in eq. (11) to include indices of type 5 and 6 corresponding to the scalar fields \( \Phi_5 \) and \( \Phi_6 \). This leads to the same result for the regularized contribution to the three-point function.
On the other hand, the method based on using appropriate tensors as the starting point for the regularization, works neatly in the $AdS_5 \times S^5$ case as well and, in addition, allows us to understand why certain spherical harmonics fail to regularize the three-point function. The spherical harmonic (26) corresponds to the tensor\(^3\)

\[
\Delta/2 \quad \frac{1}{1!} \frac{1}{2} \frac{1}{3} \frac{1}{k} = 1, \\
\Delta/2 \quad \frac{1}{1!} \frac{1}{2!} \frac{1}{3!} \frac{1}{k} = (-1)^k, 
\]

(28)

As in the $AdS_5 \times S^5$ case one can regularize the ill defined contribution to the three-point function by dressing the tensor above with extra indices corresponding to fields not already present in the chiral primary (26) and sending the number of extra indices to zero at the end of the calculation. A choice for the extension of the tensor indices which mimics closely the one of the $AdS_5 \times S^5$ is

\[
\Delta/2 \quad \frac{1}{1!} \frac{1}{2!} \frac{1}{3!} \frac{1}{4!} \frac{1}{k} = (-1)^k, \\
\Delta/2 \quad \frac{1}{1!} \frac{1}{2!} \frac{1}{3!} \frac{1}{4!} \frac{1}{k} = (-1)^k. 
\]

(29)

where symmetrization is understood. This tensor corresponds to the spherical harmonic

\[
Y_{\Delta} = \tilde{N}_{\Delta} T^{\Delta} e^{i X \Delta/2} e^{i \omega_1} F_1 \left( -l, -l; 1; e^{2 \omega_1} r^2 \right). 
\]

(30)

Using this spherical harmonic instead of $Y_\Delta$ from eq. (26) when evaluating the divergent part of (24) and (25) and sending $l$ to zero at the end of the calculation leaves one with the following contribution to the three-point function

\[
C_{\text{3 regularized}} = -\frac{\Delta + 2}{\Delta} C_{\text{3 finite}}, 
\]

(31)

so that the full three-point function in the $AdS_5 \times \mathbb{C}^3$ case becomes\(^4\)

\[
C_{\text{total}} = \frac{1}{N} \left( \frac{\lambda}{2 \pi i} \right)^{1/4} (2k) (2\pi)^{\Delta} \left( \frac{2\sqrt{2} \Delta + 1}{\Delta} \right). 
\]

(32)

A natural question to ask is what would happen if we chose an even more general tensor as a regulator such as the tensor

\[
\Delta/2 \quad \frac{1}{1!} \frac{1}{2!} \frac{1}{3!} \frac{1}{4!} \frac{k_1}{k_2} \frac{k_3}{k_4} = (-1)^{k_2+k_4}, \\
\Delta/2 \quad \frac{1}{1!} \frac{1}{2!} \frac{1}{3!} \frac{1}{4!} \frac{k_1}{k_2} \frac{k_3}{k_4} = (-1)^{k_2+k_4}, 
\]

(33)

where $\sum_{i=1}^4 k_i = l$. For such tensors it is possible to show that the regularization procedure leads to negative powers of $\alpha$ in the three-point function unless $k_1 = k_2$. Negative powers of $\alpha$ in the three-point function imply a divergence of the three-point function in the maximal limit and the correlator is thus not fully regularized. Enforcing $k_1 = k_2$ is not possible to construct a traceless symmetric tensor unless $k_1 = k_2 = 0$ which brings us back to the previous case (29).

\[\text{(35)}\]

\[\text{The chiral primary operators of ABJM can be written in the form (C)}_{\Delta/2}^{1 \cdots 1 \cdots 1} \left( Z_k \right)^{i_1 \cdots i_2} \left( Z^{i_3 \cdots i_4} \right) \text{ where the } Z_k \text{ ’s can be any of the four complex fields of the theory and where } C \text{ is symmetric in upper and lower (independently) in lower indices and the trace taken over any pair of upper and lower (individually) in lower indices vanishes.}\]

\[\text{(37)}\]

\[\text{We have left out a sign here since three-point functions are anyway only determined up to a phase factor.}\]