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Integration rules for loop scattering equations

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Abstract: We formulate new integration rules for one-loop scattering equations analogous to those at tree-level, and test them in a number of non-trivial cases for amplitudes in scalar φ^3-theory. This formalism greatly facilitates the evaluation of amplitudes in the CHY representation at one-loop order, without the need to explicitly sum over the solutions to the loop-level scattering equations.

Keywords: Scattering Amplitudes, Duality in Gauge Field Theories, 1/N Expansion

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Contents

1 Introduction

The formalism of Cachazo, He and Yuan (CHY) [1–4] is an intriguing reformulation of quantum field theory that represents scattering amplitudes as integrals over an auxiliary coordinate space completely localized by δ-functions which impose a set of algebraic constraints referred to as the *scattering equations*. At tree-level, the auxiliary integral is performed over points $z_i \in \mathbb{P}^1$ associated with each particle, and the scattering equations (which fully localize the z_i's) correspond to

$$S_i \equiv \sum_{i \neq j} \frac{s_{ij}}{(z_i - z_j)} = 0\,, \tag{1.1}$$

for the i^{th} particle, with $s_{ij} \equiv (k_i + k_j)^2$ being the familiar Mandelstam invariants. The precise measure of integration for scattering amplitudes depends on the theory in question, but the constraints $\delta(S_i)$ always localize the integral to a sum over isolated solutions to the scattering equations (1.1). For n particles, there are $(n-3)!$ solutions to these equations. Integration measures for many theories are known, and a proof of this remarkable construction for scalar φ^3-theory and Yang-Mills theory has been given by Dolan and Goddard in ref. [5].

In practice, the summation over $(n-3)!$ solutions makes the formalism very cumbersome already at rather low multiplicity kinematics. Recently, two complementary methods were developed that circumvent this brute-force procedure and which directly produce the result of integration — that is, summing over all the solutions [6, 7]. Moreover, a direct link between individual Feynman diagrams and integrands for the CHY representation has been provided as well [8]. With this, one has complete control over the CHY construction at tree-level and is therefore ready to tackle the question of amplitudes at loop-level.

There are two obvious paths towards obtaining a scattering equation formalism valid at loop-level. With the now known map between CHY-integrands and tree-level Feynman diagrams, one could make use of generalized unitarity to reconstruct loop amplitudes out of on-shell, tree-level diagrams and use the tree-level scattering equations. A more elegant solution would build on the close connection between the CHY-formalism and string theory [9–12]. Indeed, steps in that direction were taken in ref. [13] and further developed in ref. [14], identifying field theory loops in terms of the genus expansion, as in string theory. The main, naïve stumbling block in that approach is the natural appearance of elliptic functions that, in ordinary perturbation theory, should be represented as integrals over rational functions. A breakthrough in this direction has recently been made by Geyer, Mason, Monteiro and Tourkine [15]. In the context of supergravity, they show how to reduce the problem of genus one to a modified problem on the Riemann sphere, where the analysis is essentially the same as at tree-level. They provide a conjecture for the n-point supergravity one-loop amplitude, and suggest how to generalize their result to any loop-order; they also provide a conjecture for super Yang-Mills amplitudes at one-loop.

In this paper, we generalize the analysis of ref. [15], and show how it naturally leads to a representation of one-loop amplitudes in φ^3-theory. The scalar case provides the simplest setting in which to understand the use of scattering equations at loop-level. As discussed in refs. [14, 15], the one-loop case essentially amounts to computing an n-point amplitude by means of an auxiliary $(n+2)$-point scattering amplitude involving two additional particles with momenta ℓ and $-\ell$ (that is, taken in the forward limit). Intuitively, this is not unlike representing loops using the Feynman tree theorem [16, 17], for example. However, the representation of amplitudes using the scattering equations appears quite a bit more magical as we will see below.

An essential ingredient that makes the scattering equation formalism work at loop-level is the freedom to shift what becomes loop momentum ℓ by an arbitrary constant in any individual term — a property that must be respected by the regularization framework being used.[1] This is because, as we will see, the scattering equation formalism naturally generates rather unfamiliar representations of loop integrands — involving 'propagators' that are almost exclusively linear in the loop momentum.

The loop-level scattering equations are nearly identical to those at tree-level, but with two additional particles with opposite (off-shell) momenta. As such, there are $(n+2-3)! = (n-1)!$ solutions in general. This counting differs from that of ref. [14] because we use loop-level scattering equations that differ due to regularization concerns that will be discussed in section 4. And we will find that the integration rules described in ref. [7] must be modified slightly to take into account the additional, off-shell momenta in the forward limit. The principal difference will be that for φ^3-theory, our representation explicitly removes tadpole contributions (similar to the dimensionally-regulated Feynman expansion). Although this paper is mainly concerned with scalar φ^3-theory, it is clear that the integration rules we describe can be applied to a much broader class of theories.

Our paper is organized as follows. In the next section we provide a lightning review of the scattering equation formalism, including the integration rules that permit us to evaluate terms without the explicit summation over solutions to the scattering equations.

[1]This is the case for dimensional regularization. Because the scattering equation formalism is independent of the number of spacetime dimensions, it is natural for us to use it here. See also the discussion in section 4.

In section 3 we turn to loop-level, using the recent supergravity solution of ref. [15] as a guide for inferring the correct integration measure for scalar φ^3-theory. We test this proposal in section 4 with concrete examples at one-loop.

2 Scattering equations and integration rules at tree-level

Recall that in the CHY formalism, ordered tree-level scattering amplitudes in massless φ^3-theory can be represented [2, 5] as follows:

$$\mathcal{A}_n^{(\varphi^3),\text{tree}} = \int d\Omega_{\text{CHY}} \left(\frac{1}{(z_1-z_2)^2(z_2-z_3)^2 \cdots (z_n-z_1)^2} \right). \tag{2.1}$$

Here, $d\Omega_{\text{CHY}}$ represents a universal integration measure together with the δ-function constraints which impose scattering equations (1.1) (and fully localize the integral):

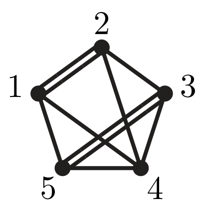
$$d\Omega_{\text{CHY}} \equiv \frac{d^n z}{\text{vol}(\text{SL}(2,\mathbb{C}))} \prod_i{}' \delta(S_i) = (z_r-z_s)^2(z_s-z_t)^2(z_t-z_r)^2 \prod_{i \in \mathbb{Z}_n \backslash \{r,s,t\}} dz_i\, \delta(S_i)\,. \tag{2.2}$$

This measure is independent of the $\text{SL}(2,\mathbb{C})$ gauge-choice of points labelled $\{r,s,t\}$. Because the δ-functions fully localize the integral (2.1), it becomes simply a sum over the $(n-3)!$ isolated solutions to the scattering equations.

Scattering amplitudes in different theories can all be represented as integrals over $d\Omega_{\text{CHY}}$, but with different integrands than that of (2.1). More generally then, we will be interested in integrals of the form:

$$\int d\Omega_{\text{CHY}}\ \mathcal{I}(z_1,\ldots,z_n)\,. \tag{2.3}$$

For the sake of concreteness, let us restrict our attention to Möbius-invariant integrals involving products of factors of the form $(z_i - z_j)$ (with $i<j$) in the denominator. We can represent integrands of this form graphically by drawing vertices for each z_i, and connecting vertices $\{z_i, z_j\}$ for each factor of $(z_i - z_j)$ appearing in the denominator. Möbius-invariance requires that each factor z_i occurs four times, resulting in integrands represented by four-regular graphs. For example, consider the integrand represented graphically by,



$$\Leftrightarrow \frac{1}{(z_1-z_2)^2(z_2-z_3)(z_3-z_4)(z_4-z_5)(z_1-z_5)(z_3-z_5)^2(z_1-z_4)(z_2-z_4)}.$$

Integration of this function $\mathcal{I}(z_1,\ldots,z_5)$ against the measure $d\Omega_{\text{CHY}}$ results in an inverse product of Mandelstam invariants — in this case, $1/(s_{12}s_{35})$.

A combinatorial rule for the result of integration for integrals of the form (2.3) was described in ref. [7], which we briefly summarize here. Integrals of this form generally result in a sum of inverse-products of multi-index Mandelstam invariants denoted $s_{ij\cdots k} \equiv$

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$s_{\{i,j,\ldots,k\}} \equiv (k_i + k_j + \cdots + k_k)^2$ (for arbitrary subsets $P \subset \{1,\ldots,n\}$). In general, each term in the sum will be a product of precisely $(n-3)$ factors,

$$\prod_{a=1}^{n-3} 1/s_{P_a}, \tag{2.4}$$

where each $P_a \subset \{1,\ldots,n\}$ denotes a subset of legs that we can always take to have at most $n/2$ elements (because $s_P = s_{P^\complement}$, with $P^\complement \equiv \mathbb{Z}_n \backslash P$, by momentum conservation). The collections of subsets $\{P_a\}$ appearing in (2.4) must satisfy the following criteria:

- for each pair of indices $\{i,j\} \subset P_a$ in each subset P_a, there are exactly $(2|P_a|-2)$ factors of $(z_i - z_j)$ appearing in the denominator of $\mathcal{I}(z_1,\ldots,z_n)$;
- each pair of subsets $\{P_a, P_b\}$ in the collection is either nested or complementary — that is, $P_a \subset P_b$ or $P_b \subset P_a$ or $P_a \subset P_b^\complement$ or $P_b^\complement \subset P_a$;

if there are no collections of $(n-3)$ subsets $\{P_a\}$ satisfying the criteria above, the result of integration will be zero.

These integration rules produce the result of the integration in eq. (2.1) for an arbitrary number of external legs in tree-level φ^3-theory. In the next section, we will need integration rules for loop integrands of one-loop with $(n+2)$ external legs, two of which are neighboring with off-shell momenta ℓ and $-\ell$. The rules will be quite similar to those described above, but with a few small changes. One prominent change will be the appearance of Mandelstam-like objects generalized to include off-shell momenta:

$$[i,j,\ldots,k] \equiv (k_i + k_j + \cdots + k_k)^2 - (k_i^2 + k_j^2 + \cdots + k_k^2)\,. \tag{2.5}$$

Notice that $[i,j,\ldots,k]$ becomes identical to $s_{ij\cdots k}$ when all the momenta are on-shell and massless.

3 Scattering equations for one-loop amplitudes

The scattering equations at one-loop-level given in ref. [15] provide a great simplification over the ones considered in refs. [13, 18, 19]. We refer to those references for details.

At tree-level, the scattering equations are defined on the Riemann surface as discussed above. The locations of the external legs are parametrized by the coordinates, z_i, where i runs from 1 to n for the n-point amplitude. At one-loop level one has to consider scattering equations on the torus — the genus-one surface. Here, τ and z parametrize the torus, and the points z_i has the same meaning as in the tree-level case, i.e., they are the positions of the external legs. At one-loop the scattering equations are

$$\operatorname*{Res}_{z_i} P(z,z_i|q)^2 = 2k_i \cdot P(z=z_i,z_i|q) = 0\,, \qquad P(z=z_0,z_i|q)^2(z_0) = 0\,, \tag{3.1}$$

where z_0 is an arbitrary point on the torus and the one-form $P(z,z_i|q)$ is the solution to the following differential equation

$$\bar\partial P(z,z_i|q) = 2\pi i \sum_i^n k_i \bar\delta(z-z_i)dz\,. \tag{3.2}$$

The solution can be parametrized by

$$P(z, z_i|q) = 2\pi i\ell dz + \sum_i^n k_i \left(\frac{\theta_1'(z-z_i)}{\theta_1(z-z_i)} + \sum_{j\neq i} \frac{\theta_1'(z_{ij})}{n\theta_1(z_{ij})} \right) dz\,, \tag{3.3}$$

on the torus where q is related to the modular variable τ in the following way: $q = e^{2\pi i\tau}$. ℓ will turn out to play the role of the loop momentum. $\theta_1(z)$ is the standard modular function that also appears in string theory.

The one-form $P(z, z_i|q)$ can be greatly simplified in the limit $q = e^{2\pi i\tau} \to 0$, where $\tau \to +i\,\infty$, and by changing variables from z_i to σ_i and z to σ using the following redefinitions: $\sigma_i = e^{2\pi i(z_i - \tau/2)}$, $\sigma = e^{2\pi i(z-\tau/2)}$. In the new variables translational invariance of z becomes scaling invariance of σ, (i.e. $dz = \frac{d\sigma}{2\pi i\sigma}$), and in the limit one observes that

$$\frac{\theta_1'(z-z_i)}{\theta_1(z-z_i)} dz \to \frac{-d\sigma}{2\sigma} + \frac{d\sigma}{\sigma - \sigma_i}\,. \tag{3.4}$$

Using momentum conservation $(\sum_i^n k_i)\frac{-d\sigma}{2\sigma} = 0$ in the limit yields

$$P(z, z_i|q) \to P(\sigma, \sigma_i) = \ell\frac{d\sigma}{\sigma} + \sum_i^n \frac{k_i d\sigma}{\sigma - \sigma_i}\,, \tag{3.5}$$

after redefining $\ell \to \ell - \sum_{i<j}^n (k_i - k_j)\cot(\pi z_{ij})\frac{1}{2in}$. We now find that

$$P(\sigma, \sigma_i)^2 - \frac{\ell^2\, d\sigma^2}{\sigma^2} = \sum_i^n \frac{2\ell\cdot k_i\, d\sigma^2}{\sigma(\sigma-\sigma_i)} + \sum_{i<j}^n \frac{2k_i\cdot k_j\, d\sigma^2}{(\sigma-\sigma_i)(\sigma-\sigma_j)}\,. \tag{3.6}$$

The combination $P(\sigma, \sigma_i)^2 - \frac{\ell^2 d\sigma^2}{\sigma^2}$ has only single poles. It is easy to calculate the residues of these single poles and they are

$$\mathcal{S}_i \equiv \frac{[\ell, k_i]}{\sigma_i} + \sum_{j\neq i}^n \frac{[i,j]}{(\sigma_i - \sigma_j)}\,, \tag{3.7}$$

for the single pole at σ_i and

$$\mathcal{S}_0 \equiv \sum_i^n \frac{[\ell, i]}{-\sigma_i}\,, \tag{3.8}$$

for the single pole at $\sigma = 0$. The residue of $\sigma = \infty$ is zero. It is easy to check that $\sum_{i=1}^n \mathcal{S}_i = -\mathcal{S}_0$. Furthermore, $\sum_{i=1}^n \sigma_i\mathcal{S}_i = 0$. The equations defined by $\mathcal{S}_0 = 0$ and $\mathcal{S}_i = 0$ are the one-loop scattering proposed in [15] on the Riemann sphere, with ℓ playing the role of the loop momentum. As shown above only $(n-1)$ of these equations are independent. If we compare them with the tree scattering equations, it is clear that the one-loop scattering equations for n-point amplitudes are very similar to the tree-level scattering equations for $(n+2)$ external legs, where two legs of off-shell momenta $\ell, -\ell$ have been inserted and fixed to the values $\sigma_\ell = 0$ and $\sigma_{-\ell} = \infty$. To avoid confusion we will distinguish the tree-level case from the one-loop case by using z_i for the insertions at tree-level and σ_i for the insertions at one-loop level. One crucial difference between the tree-level case and the one-loop case is that we take ℓ and $-\ell$ to be off-shell.

Since two points $0, \infty$ have been fixed ($\sigma_\ell = 0$ and $\sigma_{-\ell} = \infty$), the general $\mathrm{SL}(2,\mathbb{C})$-transformation on the Riemann sphere $\frac{a\sigma+b}{c\sigma+d}$ is reduced to just $\frac{a\sigma}{d}$. This means that we are in the one-loop case just left with a scaling invariance, which, using $ad-bc=1$ reads $\sigma \to a^2\sigma$. The scaling invariance can be immediately observed in the scattering equations (3.7) and can also be understood from the definition $\sigma = e^{2\pi i(z-\tau/2)}$. The scaling symmetry in the σ_i coordinates corresponds to translational invariance in the original one-loop torus variables.

Our goal now is to find the correct CHY measure at loop-level for color ordered φ^3 theory, insisting on the scaling invariance discuss above. We will start the discussion by recalling the tree-level measure

$$d\Omega_{\mathrm{CHY}} \equiv \frac{d^n z}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))} \prod_i{}' \delta(S_i) = (z_r - z_s)^2 (z_s - z_t)^2 (z_t - z_r)^2 \prod_{i \in \mathbb{Z}_n \backslash \{r,s,t\}} dz_i\, \delta(S_i)\,. \tag{3.9}$$

Introducing $z_{ij} \equiv (z_i - z_j)$, we can write tree-level amplitudes in the following general form

$$\int \left(\prod_{i=1}^n dz_i\right) \left(z_{rs} z_{st} z_{tr} \prod_{a \neq r,s,t} \delta\left(S_a\right)\right) \left(\frac{1}{\mathcal{F}(z)}\right) \left(\frac{1}{d\omega}\right)\,. \tag{3.10}$$

Now let us analyze the four factors in (3.10). Since we have only $(n-3)$ independent scattering equations, we correspondingly insert only $(n-3)$ δ-function constraints. However, the result must be independent of the choice of which equations we choose. This independence is precisely achieved by the factor $z_{rs}z_{st}z_{tr}$ that is inserted in the measure and which renders the combined expression permutation invariant. This factor provides also the same transformation under the $\mathrm{SL}(2,\mathbb{C})$ group as that of the three scattering equations that have been removed. Because of these first two factors in eq. (3.10), $\mathcal{F}$ must transform as

$$\mathcal{F}(z) \to \left(\prod_{i=1}^n \frac{(ad-bc)^2}{(cz_i+d)^4}\right) \mathcal{F}(z)\,, \tag{3.11}$$

under the $\mathrm{SL}(2,\mathbb{C})$ transformation

$$z_i \to \frac{az_i + b}{cz_i + d}\,. \tag{3.12}$$

Different choices of this factor $\mathcal{F}$ with proper transformation properties will define different theories. The last factor $d\omega \equiv \frac{dz_r dz_s dz_t}{z_{rs}z_{st}z_{tr}}$ provides the Koba-Nielsen gauge fixing.

Having understood how the integrand is composed for a tree-level amplitude in the CHY formalism, we now proceed to deduce the corresponding integrand at one-loop level. First, since there are now only $(n-1)$ independent loop scattering equations, we can have only $(n-1)$ δ-function constraints $\delta(\mathcal{S}_i)$. Again, to make the result independent of the choice of which equation we eliminate, we need to insert a factor with the same scaling property as the δ-function we removed. A natural combination is $\left(\sigma_l \prod_{j\neq l}^n \delta(\mathcal{S}_j)\right)$.[2] Now

[2] The same choice can also be inferred from the corresponding factor at tree-level: the term $z_{ij}z_{jk}z_{ki}$ with $z_{i=\ell} = 0$ and $z_{j=-\ell} = \infty$ reduces to $z_{ki} = z_k$.

(in a similar way to the tree-level case) we can write down the proposed integration at one-loop level

$$\int \frac{1}{\text{vol}(GL(1))} \left(\prod_{i=1}^{n} d\sigma_i\right) \left(\sigma_l \prod_{j\neq l}^{n} \delta(\mathcal{S}_j)\right) \left(\frac{1}{\mathcal{F}(\sigma_i)}\right) . \tag{3.13}$$

Scaling invariance now requires that $\mathcal{F}(\lambda\sigma_l) = \lambda^{2n}\mathcal{F}(\sigma_l)$. Using the standard Faddeev-Popov method, we can gauge fix any σ_k to a fixed value. We will call this the (k,l) gauge-choice, where l is the scattering equations removed and k is the σ_k that has been fixed. With this gauge choice eq. (3.13) reads

$$\int \left(\prod_{i=1}^{n} d\sigma_i\right) \left(\sigma_l \prod_{j\neq l}^{n} \delta(\mathcal{S}_j)\right) \left(\frac{1}{\mathcal{F}(\sigma_i)}\right) \left(\frac{1}{d\omega}\right), \qquad d\omega = \frac{d\sigma_k}{\sigma_k} . \tag{3.14}$$

Next we will consider the possible choices of $\mathcal{F}(\sigma_i)$ corresponding to different theories, such as gravity, Yang-Mills theory, and scalar field theory at one-loop level.

For gravity there is no color ordering, the amplitude must be symmetric in the external legs and we therefore require that $\mathcal{F}(\sigma_i)$ is totally permutation invariant. The scaling degree $2n$ leads to the natural choice $\mathcal{F}(\sigma_i) = \mathcal{I}^{-1}G^2$ with $G = \prod_{i=1}^{n}\sigma_i$ and $\mathcal{I}^{-1}$ being a scale invariant expression. An example for $\mathcal{I}$ in supergravity has been conjectured in ref. [15] with the gauge fix $(k,l) = (1,1)$.

For Yang-Mills theory, ref. [15] conjectured the following factor to go into the expression for $\mathcal{F}(\sigma_i)$

$$PT_n(\gamma) = \frac{\sigma_{\ell(-\ell)}}{\sigma_{\ell\gamma(1)}\sigma_{\gamma(1),\gamma(2)}\cdots\sigma_{\gamma(n-1),\gamma(n)}\sigma_{\gamma(n)(-\ell)}} , \tag{3.15}$$

where γ is an element of the n-point permutation group $\mathfrak{S}_n$. We will exclusively be considering equations where $\sigma_\ell = 0$ and $\sigma_{-\ell} = \infty$, in which case the PT factor simplifies to

$$PT_n(\gamma) = \frac{1}{\sigma_{\gamma(1)}\sigma_{\gamma(1)\gamma(2)}\cdots\sigma_{\gamma(n-1)\gamma(n)}} . \tag{3.16}$$

Since the scaling degree of PT is n, we need another factor in $\mathcal{F}(\sigma_i)$ with scaling degree n in order to arrive at the overall scaling of degree $2n$. It is natural to assume that the other factor is G, defined above. Thus, for a given color ordering γ we should expect $\mathcal{F}_\gamma(\sigma_i) = \mathcal{I}^{-1}PT_n(\gamma)\,G$ where $\mathcal{I}^{-1}$ again is a scale invariant expression. After taking the gauge fixing $(k,l) = (1,1)$, we arrive at the expression in ref. [15]. A possible $\mathcal{I}$ for super Yang-Mills theory has been conjectured in ref. [15].

Now we will concern ourselves with the scalar case. Having gained experience from the supergravity and super Yang-Mills theory cases, it is natural to assume that for color ordered bi-adjoint scalar φ^3-theory, we should have $\mathcal{F}(\sigma_i) = PT_n(\gamma_1)PT_n(\gamma_2)$ with γ_1, γ_2 being two permutations in $\mathfrak{S}_n$. This assumption arises from an analogy with the tree-level case in ref. [2], where the gluon and then the bi-adjoint scalar amplitude is obtained from the graviton amplitude via the following substitutions in the integrand:

$$
\begin{aligned}
(\mathrm{Pf}'\Psi)^2 &\to \mathrm{Pf}'\,\Psi\frac{1}{\sigma_{\gamma_1(1)\gamma_1(2)}\cdots\sigma_{\gamma_1(n)\gamma_1(1)}} \\
&\to \frac{1}{\sigma_{\gamma_1(1)\gamma_1(2)}\cdots\sigma_{\gamma_1(n)\gamma_1(1)}}\frac{1}{\sigma_{\gamma_2(1)\gamma_2(2)}\cdots\sigma_{\gamma_2(n)\gamma_2(1)}}\,.
\end{aligned}
$$

In other words, the naïve expectation would be for the one-loop scalar amplitude $\mathcal{A}$ to be given by

$$
\mathcal{A}(\gamma_1|\gamma_2) \equiv \int \frac{\prod_{i=1}^n d\sigma_i}{d\sigma_k}\sigma_l\sigma_k\prod_{j\neq l}^n \delta(\mathcal{S}_j)PT_n(\gamma_1)PT_n(\gamma_2)\,. \tag{3.17}
$$

The analogue quantity of $\mathcal{A}(\gamma_1|\gamma_2)$ at tree-level is $m(\gamma_1|\gamma_2)$ in ref. [3], which is nothing but the inverse of the momentum kernel $\mathcal{S}[\gamma_1|\gamma_2]$ that was first defined in [20–22]. We thus have $\mathcal{S}[\gamma_1|\gamma_2] = m(\gamma_1|\gamma_2)^{-1}$, with

$$
\mathcal{S}[i_1,\ldots,i_k|j_1,\ldots,j_k] = \prod_{t=1}^k\left(s_{i_t1} + \sum_{q>t}^k \Theta(i_t,i_q)s_{i_t,i_q}\right), \tag{3.18}
$$

where Θ is the Heaviside function. The function $\mathcal{A}$ at loop-level can be thought of as the inverse (one-loop) momentum kernel.

However it turns out that the naïve choice for $\mathcal{A}$ above is not yet complete. Firstly, as in the tree-level case, to get the scalar amplitude with colour ordering γ from the bi-adjoint amplitude, we must set $\gamma_1 = \gamma_2 = \gamma$. With this ordering, the two extra legs k_l and k_{-l} will have been inserted between legs $k_{\gamma(1)}$ and $k_{\gamma(n)}$. The two extra legs do not correspond to physical external states, but can be considered as appearing when a loop is opened up in a Feynman diagram by cutting a one-loop propagator. Since we can cut any loop propagator, this physical picture suggests that to get the complete one-loop integrand of a given color ordering, we should sum over all cyclic orderings. In other words, the pair $\{\ell,-\ell\}$ should be inserted at all possible places of the given color ordering of n-points. From this we are now led to the correct compact expression:

$$
\mathcal{A}_{\varphi^3}(\gamma) \equiv (-1)^n\int\frac{d^d\ell}{\ell^2}\int\frac{\prod_{i=1}^n d\sigma_i}{d\sigma_k}\sigma_l\sigma_k\prod_{j\neq l}^n\delta(\mathcal{S}_j)\sum_{\text{cyclic}}(PT_n(\gamma))^2\,, \tag{3.19}
$$

Having obtained this proposal (3.19) for one-loop scalar amplitudes, we now use the δ-function constraints to integrate out the σ_i's. Using (3.7), it is straightforward to find the elements of the Jacobian,

$$
\begin{aligned}
\frac{\partial\mathcal{S}_i}{\partial\sigma_j} &= \frac{[i,j]}{(\sigma_i-\sigma_j)^2}, \qquad i\neq j\,, \\
\frac{\partial\mathcal{S}_i}{\partial\sigma_i} &= -\frac{[\ell,i]}{\sigma_i^2} - \sum_{j\neq i}\frac{[i,j]}{(\sigma_i-\sigma_j)^2}\,.
\end{aligned} \tag{3.20}
$$

Putting all these pieces together, we finally arrive at

$$
\mathcal{A}_{\varphi^3}(\gamma) = (-1)^n\int\frac{d^d\ell}{\ell^2}\sum_{\text{cyclic}}\sum_{\text{solutions}}\frac{\sigma_l\sigma_k}{(-)^{l+k}\mathcal{J}(\mathcal{S})_l^k}(PT_n(\gamma))^2\,, \tag{3.21}
$$

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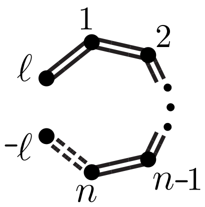

Figure 1. The CHY graph for n-points with ordering $\{1, 2, \ldots, n\}$. The dashed line between $n, -\ell$ disappears because $\sigma_{-\ell} = \infty$.

where the $\mathcal{J}(\mathcal{S})_l^k$ is the determinant of Jacobian matrix after deleting the l-th row and k-th column, and the sum runs over the solutions to the loop-level scattering equations. Although there is also a sum over cyclic permutations of γ in eq. (3.21), we need to calculate only one set, obtaining the others trivially by relabelling.

Just as at tree-level, we can associate a CHY graph with the one-loop integrand $(PT_n(\gamma))^2$ in (3.21). Such a one-loop graph for the integrand is illustrated in figure 1. The graph is very similar to the CHY graph for the full tree-level scalar $(n+2)$-point amplitude, because the CHY integral in equation (3.19) can be interpreted as the $(n+2)$-point tree level amplitude with gauge choice $\sigma_{n+1} = \infty$, $\sigma_{n+2} = 0$. A point gauge-fixed to infinity makes no explicit appearance when carrying out CHY integrals, but in CHY graphs one should never the less also draw lines for factors that disappear upon gauge-fixing. For this reason the graph in figure 1 retains the lines between points n and $-l$. When so drawn, the integration rules of ref. [7] can immediately be applied to one-loop CHY graphs with two minor modifications. The final result can still be presented in the form of eq. (2.4), which will provide the full result of the integration in (3.21) without explicitly solving the one-loop scattering equations and summing over all of them. The two modifications are the following. First, instead of having poles $\frac{1}{s_{P_a}}$, we must replace them by $\frac{1}{[P]}$ where the notation $[P]$ has been defined by eq. (2.5). In the massless case, the two expressions are the same, but for off-shell momenta with $\ell^2 \neq 0$, they are different. Secondly, we should explicitly exclude the set $P = \{\ell, -\ell\}$ (or its complement),[3] and it is for this reason that no lines have been drawn between points l and $-l$ in figure 1. Not including the set $P = \{\ell, -\ell\}$ eliminates diagrams with singular zero-momentum propagators associated with tadpoles. As a side remark we would like to note that it is also possible to write up the specific individual Feynman diagrams at loop-level; such a decomposition will be similar to an n-gon decomposition into triangle diagrams as was considered in ref. [8].

4 Scalar one-loop amplitude examples

In this section, we will demonstrate that the results obtained by solving the one-loop scattering equations using the integration measure proposed above match those obtained from the Feynman diagram expansion at one-loop order, after the proper regularization of the singular terms associated with zero momentum propagation. Furthermore, these

[3]Obviously, a set P with only one element (or its complement) should not be included, neither at tree-level nor at the one-loop level.

Figure 2. The Feynman diagram at two points.

results can be obtained directly from the associated loop-level CHY graph using our loop-level integration rules.

We will start with the one-loop integrand for the two-point 'amplitude' of φ^3-theory. Although this example is quite singular, it is simple enough to demonstrate many features of our calculation. In particular, the augmented four-point amplitude with two additional external legs ℓ and $-\ell$ is well defined and is in fact the simplest example to start with. We will first present the calculation in terms of Feynman diagrams, then explicitly use the scattering equations, and finally present the corresponding CHY graph and the result of employing the loop integration rule.

Using Feynman diagrams: without considering the tadpole diagram, there is only one term in the one-loop integrand,

$$\frac{1}{\ell^2(\ell+k_1)^2}\,, \tag{4.1}$$

corresponding to the diagram

Using the general partial fraction formula

$$\frac{1}{\prod_{i=1}^{n} D_i} = \sum_{i=1}^{n} \frac{1}{D_i \prod_{j\neq i}(D_j - D_i)}\,, \tag{4.2}$$

that was also exploited in ref. [15], we can split the integrand into

$$\frac{1}{\ell^2(2\ell\cdot k_1)} + \frac{1}{(\ell+k_1)^2(-2\ell\cdot k_1)} = \frac{1}{\ell^2(2\ell\cdot k_1)} + \frac{1}{\widetilde{\ell}^2(2\widetilde{\ell}\cdot k_2)}\,, \tag{4.3}$$

where we have used the on-shell condition $k_1^2 = 0$ and defined the variable $\widetilde{\ell} = \ell + k_1$ for the second term. Since, with a proper regularization (such as dimensional regularization), we can freely shift the loop momentum, we can identify $\widetilde{\ell} = \ell$ in the second term of (4.3) and write

$$\frac{1}{\ell^2[\ell,1]} + \frac{1}{\ell^2[\ell,2]}\,. \tag{4.4}$$

In fact, using that $k_2 = -k_1$ we now see that the sum in (4.4), the integrand of the on-shell bubble diagram, adds up to zero. This assumes that the integration really has been properly regularized so that the shift is allowed. Around $d = 4$ dimensions the massless φ^3 theory we are considering suffers from both ultraviolet and infrared divergences. Also, a mass term is not protected, and is thus expected to be generated in this theory at loop level from precisely this kind of two-point function: the infrared divergences already give a strong hint that such a mass generation will occur. Indeed, in this theory a massless on-shell particle can decay into two in the forward direction by the self-interaction, thus making the very definition of the S-matrix of the exactly massless theory subtle at the quantum level [23]. It is probably best to consider the massless theory only around $d = 6$ dimensions, where it is classically scale invariant and perturbatively renormalizable.

Let us also emphasize some points about the result (4.4). First, the two terms are related to each other by $\mathbb{Z}_2$ cyclic permutation. As we will see, this is a general feature. Secondly, although they sum up to be zero, each term will appear in different orderings $PT_n(\gamma)$ when we use the scattering equations. Thus it is necessary to write them in the form shown in (4.4). A similar phenomenon occurs in all later examples.

Using the one-loop scattering equations: to use the setup presented in the previous section, we need to make a gauge choice (k, l), i.e. choose which scattering equation $\mathcal{S}_l$ is to be removed and which variable σ_k is to be fixed. However, when we do this in this two-point example (a highly singular case), a subtle point appears. The reason is the following. After using momentum conservation, the two scattering equations become (keeping $k_1^2 \neq 0$ as regulator at the intermediate level of calculations)

$$\mathcal{S}_1 = \frac{[\ell, 1]}{\sigma_1} + \frac{[1, 2]}{\sigma_1 - \sigma_2} = 0\,, \qquad \mathcal{S}_2 = \frac{[\ell, 2]}{\sigma_2} + \frac{[1, 2]}{\sigma_2 - \sigma_1} = 0\,. \tag{4.5}$$

This leads to the identity $\frac{[\ell,1]}{\sigma_1} = \frac{[\ell,1]}{\sigma_2}$. Thus for general $\ell \cdot k_1 \neq 0$, we arrive at $\sigma_1 = \sigma_2$. In other words, we cannot gauge fix $\sigma_1 = 1$ and leave σ_2 to be a free variable. Thus we have to introduce another type of regulator μ:

$$\mathcal{S}_1 = \frac{[\ell, 1]}{\sigma_1} + \frac{\mu}{\sigma_1 - \sigma_2} = 0\,, \qquad \mathcal{S}_2 = -\frac{[\ell, 1]}{\sigma_2} - \frac{\mu}{\sigma_2 - \sigma_1} = 0\,. \tag{4.6}$$

Because of the special (singular) kinematics associated with the pair $\{\ell, -\ell\}$ that introduces on-shell bubbles (we denote bubbles on-shell or off-shell depending on the nature of their external legs), to arrive at well defined results, we need to sum over cyclic orderings before we remove the regularization.

Choosing the color ordering $\gamma = \{1, 2\}$ and taking the gauge choice $(k, l) = (1, 1)$, we get for the integrand

$$\frac{-1}{\mu} + \frac{-1}{-\mu + [\ell, 2]}\,, \tag{4.7}$$

Similarly the same gauge choice for the color ordering $\gamma = \{2, 1\}$ will lead to

$$\frac{1}{\mu} + \frac{1}{[\ell, 2]}\,. \tag{4.8}$$

We see that adding these two terms together and carefully taking the limit $\mu \to 0$, we get again a zero result as in eq. (4.4).

Interpretation via a CHY graph: we now present the corresponding CHY graph given by $PT(\gamma)^2$. For the ordering $\gamma = \{1, 2\}$, the graph is the following: we have four ordered nodes $\{\ell, 1, 2, -\ell\}$, and their connections are $\{(\ell, 1)_2, (1, 2)_2\}$. Here we have used subscript

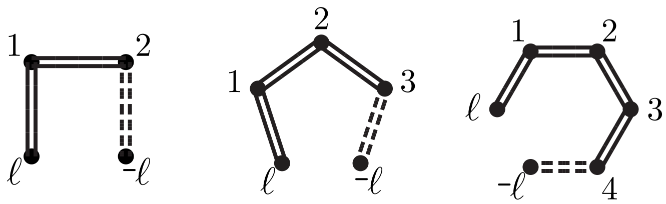

Figure 3. The CHY graphs for two-point (left), three-point (middle) and four-point(right).

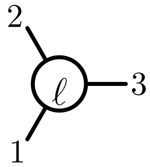

Figure 4. The triangle contribution at three points.

to indicate how many lines connect two nodes (see figure 3). Using the tree mapping rule, naïvely, we get following possible poles: $\frac{1}{[\ell,1]}$, $\frac{1}{[1,2]}$. However, the complement of the pole $\frac{1}{[1,2]}$ is $\frac{1}{[-\ell,\ell]}$ which has been removed explicitly in the definition of CHY diagram (i.e., there is no such denominator in the integrand $(PT_2)^2$), so we should not include it. This is the modification of the integration rule we need when it is applied at one-loop level. Thus we are left with only the pole $1/[\ell,1]$, which gives the final expression $\frac{1}{\ell^2[\ell,1]}$. Including the other cyclic permutation, we end up with the same result as using the scattering equations.

Having done the two point example, we will next move on to the next simplest thing, the one-loop integrand of the color ordered three-point amplitude.

Using Feynman diagrams: for the color-ordered integrand of amplitude $A(1,2,3)$, there is one triangle and three on-shell bubbles related by $\mathbb{Z}_3$ cyclic symmetry.[4] The triangle is given by

$$\begin{aligned} T_{3;(1|2|3)} &= \frac{1}{\ell^2(\ell+k_1)^2(\ell-k_3)^2} = \frac{1}{(2\ell\cdot k_1)(-2\ell\cdot k_3)\ell^2} \\ &\quad + \frac{1}{(-2\ell\cdot k_1)(-2\ell\cdot k_1-2\ell\cdot k_3)(\ell+k_1)^2} + \frac{1}{(2\ell\cdot k_3)(2\ell\cdot k_1+2\ell\cdot k_3)(\ell-k_3)^2} \\ &= \frac{1}{\ell^2[\ell,1][3,-\ell]} + \frac{1}{\ell^2[\ell,2][1,-\ell]} + \frac{1}{\ell^2[\ell,3][2,-\ell]}\,, \end{aligned} \tag{4.9}$$

where from the second to the third equation, we have used a shift of momentum ℓ, which of course is valid only under the integration. It is easy to see that these three terms are related by $\mathbb{Z}_3$ cyclic permutations. Similarly we can split the three on-shell bubbles that are related by cyclic ordering. A typical one is[5]

$$T_{2;(1|23)} = \frac{1}{\ell^2(\ell+k_1)^2 s_{23}} = \frac{1}{\ell^2[\ell,1][2,3]} + \frac{1}{\ell^2[2,3][1,-\ell]}\,. \tag{4.10}$$

[4]Again we will not include the tadpole diagrams.

[5]For an on-shell amplitude, we will have $s_{23}=0$. Thus to have a well defined meaning, one should regularize $k_i^2\neq 0$ for the legs $i=1,2,3$.

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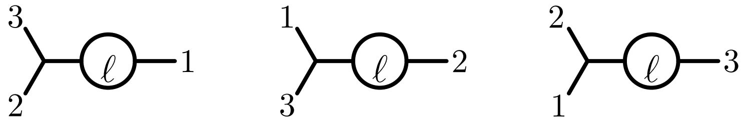

Figure 5. The three bubble contributions at three points.

To compare with the results from scattering equations and CHY graphs, we reorganize all $1\times 3+3\times 2=9$ terms into three groups, which are related to each other by $\mathbb{Z}_3$ cyclic permutations. The first group is

$$\mathcal{G}_1^{(3p)}=\frac{1}{\ell^2[\ell,1][3,-\ell]}|_{T_{3;(1|2|3)}}+\frac{1}{\ell^2[\ell,1][2,3]}|_{T_{2;(1|23)}}+\frac{1}{\ell^2[1,2][3,-\ell]}|_{T_{2;(3|12)}}\,, \tag{4.11}$$

where we have used the subscript to indicate where this term comes from. In fact, as we will see, $\mathcal{G}_1^{(3p)}$ is given by the CHY graph with ordering $\gamma=\{1,2,3\}$. Again summing over three cyclic permutations, the on-shell bubble part cancels and we are left with only the triangle contribution.

Using the scattering equations: we now use the scattering equations to find the integrand. Let us start with ordering $\gamma=\{1,2,3\}$. As expected, one will get contributions from the on-shell bubbles $(1|2+3)$ as well as $(1+2|3)$. To regulate the solutions we set $k_1^2\neq 0$ and $k_3^2\neq 0$. For the gauge choice $(k,l)=(1,1)$ we get

$$\begin{aligned}&\frac{-\ell\cdot k_2}{4(k_1\cdot k_2)(-\ell\cdot k_1+k_1\cdot k_3)(\ell\cdot k_3-k_3^2)}\\&\quad=\frac{1}{4(k_1\cdot k_2)}\left(\frac{1}{(-\ell\cdot k_1+k_1\cdot k_3)}+\frac{-1}{(\ell\cdot k_3-k_3^2)}+\frac{(k_1\cdot k_2)}{(-\ell\cdot k_1+k_1\cdot k_3)(\ell\cdot k_3-k_3^2)}\right).\end{aligned} \tag{4.12}$$

Taking the limit of $k_1^2,k_3^2\to 0$ we get

$$\frac{1}{[\ell,1][2,3]}+\frac{1}{[1,2][3,-\ell]}+\frac{1}{[\ell,1][3,-\ell]}\,, \tag{4.13}$$

which, when inserting the $1/\ell^2$-factor, is the same as $\mathcal{G}_1^{(3p)}$ in (4.11).

In the three point case having done the ordering $\gamma=\{1,2,3\}$, we should add the other two orderings $\gamma=\{3,1,2\}$ and $\gamma=\{2,3,1\}$ related by cyclic permutations. Summing all three contributions we match the Feynman expansion independently of the gauge.

Interpretation via a CHY graph: we now present the corresponding CHY graph derivation given by the integrand with $\gamma=\{1,2,3\}$: with the ordering of nodes $\{\ell,1,2,3,-\ell\}$, thus the connections are $\{(\ell,1)_2,(1,2)_2,(2,3)_2\}$ (see figure 3). Using the mapping rule, we have the following possible poles (again, since $\frac{1}{[1,2,3]}=\frac{1}{[-\ell,\ell]}$ we do not include these poles)

$$\frac{1}{[\ell,1]},\quad\frac{1}{[1,2]},\quad\frac{1}{[2,3]},\quad\frac{1}{[\ell,1,2]}\,. \tag{4.14}$$

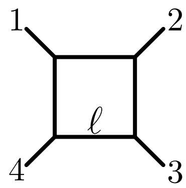

Figure 6. The box contribution $T_{4;(1|2|3|4)}$ at four points.

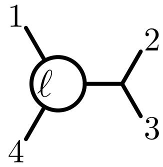

Figure 7. One of triangle contributions $T_{3;(1|23|4)}$ at four points.

Taking the compatible combinations we get the following result for the propagators

$$\frac{1}{[\ell,1][2,3]}, \qquad \frac{1}{[\ell,1,2][\ell,1]}, \qquad \frac{1}{[\ell,1,2][1,2]}\,. \tag{4.15}$$

Thus we have exactly the contribution $\mathcal{G}_1^{(3p)}$. Again adding the cyclic permutations we arrive at the complete answer.

The four-point amplitude is the first non-trivial example where we can really test the formalism. Again we will employ three different paths to get the result, and compare them.

Using Feynman diagrams: we first write down the color ordered one-loop integrand using Feynman diagrams. There is one box diagram

$$\begin{aligned} T_{4;(1|2|3|4)} &= \frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell-k_4)^2} \\ &= \frac{1}{\ell^2[\ell,1][\ell,1,2][4,-\ell]} + \frac{1}{\ell^2[\ell,2][4,1,-\ell][1,-\ell]} \\ &\quad + \frac{1}{\ell^2[\ell,3][1,2,-\ell][2,-\ell]} + \frac{1}{\ell^2[\ell,4][\ell,4,1][3,-\ell]}\,. \end{aligned} \tag{4.16}$$

Here we have used a momentum shift to reach the last line. Using identities such as $[\ell,1,2]=[3,4,-\ell]$ for four-point kinematics, it is easy to see that these four terms in (4.16) are related by $\mathbb{Z}_4$ cyclic permutations. Next there are four triangles related to each other by a $\mathbb{Z}_4$ cyclic permutation. As an example, we can consider the triangle contribution

$$\begin{aligned} T_{3;(1|23|4)} &= \frac{1}{\ell^2(\ell+k_1)^2(\ell-k_4)^2 s_{23}}, \\ &= \frac{1}{\ell^2[\ell,1][2,3][4,-\ell]} + \frac{1}{\ell^2[2,3][4,1,-\ell][1,-\ell]} + \frac{1}{\ell^2[\ell,4][\ell,4,1][2,3]}\,. \end{aligned} \tag{4.17}$$

For the bubbles, there are two different kinds in this four-point case: off-shell bubbles and on-shell bubbles. For the on-shell bubbles, there are four which are related by a $\mathbb{Z}_4$ cyclic

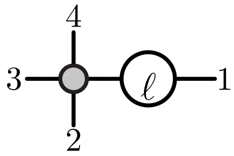

Figure 8. The bubble contribution $T_{2;(1|234)}$ at four points.

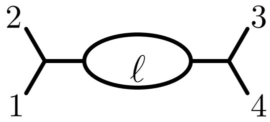

Figure 9. The bubble contribution $T_{2;(12|34)}$ at four points.

permutation. The first one is (again we use the intermediate regularization $k_i^2 \neq 0$ to make them well-defined)

$$\begin{aligned} T_{2;(1|234)} &= \frac{1}{\ell^2(\ell+k_1)^2 s_{34} s_{234}} \\ &= \frac{1}{\ell^2[\ell,1][2,3,4][3,4]} + \frac{1}{\ell^2[2,3,4][3,4][1,-\ell]} \\ &\qquad + \frac{1}{\ell^2[\ell,1][2,3][2,3,4]} + \frac{1}{\ell^2[2,3][2,3,4][1,-\ell]} \,. \end{aligned} \tag{4.18}$$

There are two off-shell bubbles. They are related by a $\mathbb{Z}_2$ permutations (i.e., $1 \to 2, 2 \to 3, 3 \to 4, 4 \to 1$). The first one is

$$T_{2;(12|34)} = \frac{1}{s_{12}\ell^2(\ell+k_{12})^2 s_{34}} = \frac{1}{\ell^2[\ell,1,2][1,2][3,4]} + \frac{1}{\ell^2[3,4][1,2][1,2,-\ell]} \,. \tag{4.19}$$

Now we reorganize all 36 terms to four groups, which are all related to each other by $\mathbb{Z}_4$ cyclic permutations. The first group is

$$\begin{aligned} \mathcal{G}_1^{(4p)} = & \frac{1}{\ell^2[\ell,1][\ell,1,2][4,-\ell]}\Big|_{T_{4;(1|2|3|4)}} + \frac{1}{\ell^2[\ell,1][2,3][4,-\ell]}\Big|_{T_{3;(1|23|4)}} \\ & + \frac{1}{\ell^2[\ell,1][\ell,1,2][3,4]}\Big|_{T_{3;(2|34|1)}} + \frac{1}{\ell^2[1,2][3,4,-\ell][4,-\ell]}\Big|_{T_{3;(4|12|3)}} \\ & + \left(\frac{1}{\ell^2[\ell,1][2,3,4][3,4]} + \frac{1}{\ell^2[\ell,1][2,3][2,3,4]}\right)\Big|_{T_{2;(1|234)}} \\ & + \left(\frac{1}{\ell^2[1,2,3][2,3][4,-\ell]} + \frac{1}{\ell^2[1,2][1,2,3][4,-\ell]}\right)\Big|_{T_{2;(4|123)}} \\ & + \frac{1}{\ell^2[\ell,1,2][1,2][3,4]}\Big|_{T_{2;(12|34)}} \,, \end{aligned} \tag{4.20}$$

where we have used the subscript to indicate where each contribution comes from.

Interpretation via a CHY graph: we now use the CHY graph procedure to reproduce the result from the Feynman diagram expansion. Again, we need to sum up four

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graphs related by $\mathbb{Z}_4$ cyclic permutation. The first one will be the graph with ordering $\{\ell,1,2,3,4,-\ell\}$ and the connections $\{(\ell,1)_2,(1,2)_2,(2,3)_2,(3,4)_2\}$ defined by corresponding PT-factor (see figure 3). We list all possible poles:

$$\begin{aligned}
\text{double-pole:}\quad & \frac{1}{[\ell,1]}\,,\quad \frac{1}{[1,2]}\,,\quad \frac{1}{[2,3]}\,,\quad \frac{1}{[3,4]}\,,\\
\text{triple-pole:}\quad & \frac{1}{[\ell,1,2]}=\frac{1}{[3,4,-\ell]}\,,\quad \frac{1}{[1,2,3]}\,,\quad \frac{1}{[2,3,4]}\,,\\
\text{quadruple-pole:}\quad & \frac{1}{[\ell,1,2,3]}=\frac{1}{[4,-\ell]}\,.
\end{aligned}\tag{4.21}$$

This yields various combinations of compatible propagators. There are five combinations containing two 2-leg poles:

$$\begin{aligned}
&\frac{1}{[\ell,1][2,3][4,-\ell]}\,, && \frac{1}{[\ell,1][2,3][2,3,4]}\,, && \frac{1}{[\ell,1][\ell,1,2][3,4]}\,,\\
&\frac{1}{[\ell,1][2,3,4][3,4]}\,, && \frac{1}{[\ell,1,2][1,2][3,4]}\,.
\end{aligned}\tag{4.22}$$

There are four combinations containing only a single 2-leg-pole:

$$\begin{aligned}
&\frac{1}{[\ell,1][3,4,-\ell][4,-\ell]}\,, && \frac{1}{[\ell,1,2][1,2][4,-\ell]}\,,\\
&\frac{1}{[1,2][1,2,3][4,-\ell]}\,, && \frac{1}{[1,2,3][2,3][4,-\ell]}\,.
\end{aligned}\tag{4.23}$$

These nine terms correspond exactly to the nine terms in $\mathcal{G}_1^{(4p)}$ (4.20).

Using scattering equations: finally, we need to produce $\mathcal{G}_1^{(4p)}$ using the scattering equations under the ordering $\gamma=\{1,2,3,4\}$. Again, to have well-defined results, we regularize it with $k_1^2\neq 0, k_4^2\neq 0$. As one can check, there are six solutions (in general $(n-1)!$ solutions for n-point). A numerical check yields the result $\mathcal{G}_1^{(4p)}$ using the gauge fix $(k,l)=(4,4)$.

Before we end this section, let us briefly discuss the number of contributions generated by CHY graphs and by Feynman diagrams. We will show that the counting with the new one-loop rules is still one-to-one, just as in the tree-level case.

For a given CHY graph, the combinations of compatible propagators that we count up are exactly those that appear in the color ordered tree-level $(n+2)$-point amplitude with extra legs l and $-l$, except that we exclude the subset $\{l,-l\}$, which corresponds to removing all Feynman diagrams associated with $\ell,-\ell$ attached to the same vertex. These Feynman diagrams correspond to the color ordered tree-level amplitude with $(n+1)$-points. Thus using the known formula for the number of color ordered diagrams in φ^3 theory with n external legs

$$C_n=\frac{2^{n-2}(2n-5)!!}{(n-1)!}\,,\tag{4.24}$$

we know immediately that each CHY graph will give $C_{n+2} - C_{n+1} = \frac{3(n-1)2^{n-1}(2n-3)!!}{(n+1)!}$ terms. When summing over cyclic orderings, we get a total number of

$$\mathcal{T}_{CHY}(n) = \frac{3n(n-1)2^{n-1}(2n-3)!!}{(n+1)!}\,. \tag{4.25}$$

On the other hand for each n-gon in the Feynman diagram expansion, after partial fractioning, we have n terms, corresponding to the n choices of opening up a single propagator. After each such opening-up of a propagator, we get a color ordered tree-level Feynman diagram with $(n+2)$-points. Different openings give different orderings, where the pair $\{\ell,-\ell\}$ is inserted between different nearby vertexes $\{i,i+1\}$. Again, the Feynman diagrams obtained this way do not contain pairs of $\ell,-\ell$ attached to the same vertex. They are again tree-level Feynman diagrams with $(n+1)$-points. Combining everything we get the counting

$$n(C_{n+2} - C_{n+1})\,, \tag{4.26}$$

which is identical to the one given in eq. (4.25).

5 Conclusion and discussion

We have shown how the diagrammatic integration rules for scattering equations that were first developed for tree-level amplitudes have an immediate extension to one-loop level. The integration rules at loop level follow from those at tree-level with the following modification: the loop CHY integrand has to be compensated so that it scales correctly. This naturally leads to valid integrands for the different kinds of theories. Here we have spelled out in great detail how the procedure does appear to produce correct integrands for scalar φ^3-theory by systematically working through the low-point cases. When considering scattering equations at loop-level it is essential to specify a regularization, and for the procedure to work we need to be able to shift loop momentum by constants in the integrand. A regularization scheme such as dimensional regularization should ensure this. Because we have only been interested in demonstrating the mechanism through which the scattering equation formalism at loop level can generate the correct set of diagrams, we have ignored all issues that arise when actually performing the loop integration. In particular, the propagators should of course be given the usual $i\epsilon$-prescription of Feynman propagators.

The procedure that we have presented seems to be generalizable to higher loops. At each loop order two more legs are added at the intermediate step. This is one obvious extension to pursue in the future.

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