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Wulff-Nilsen, Christian

Publication date:
2008

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
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Christian Wulff-Nilsen

Technical Report no. 08-07
ISSN: 0107-8283
Computing the Maximum Detour of a Plane Graph in Subquadratic Time

Christian Wulff-Nilsen *

June 25 2008

Abstract

Let \( G \) be a plane graph where each edge is a line segment. We consider the problem of computing the maximum detour of \( G \), defined as the maximum over all pairs of distinct points \( p \) and \( q \) of \( G \) of the ratio between the distance between \( p \) and \( q \) in \( G \) and the distance \( |pq| \). The fastest known algorithm for this problem has \( \Theta(n^2) \) running time where \( n \) is the number of vertices. We show how to obtain \( O(n^{3/2} \log^3 n) \) expected running time. We also show that if \( G \) has bounded treewidth, its maximum detour can be computed in \( O(n \log^3 n) \) expected time.

1 Introduction

Given a geometric graph \( G \), its stretch factor (or dilation) is the maximum over all pairs of distinct vertices \( u \) and \( v \) of the ratio between the distance between \( u \) and \( v \) in \( G \) and the Euclidean distance \( |uv| \) between \( u \) and \( v \).

A spanner is a network with small stretch factor. Spanners that keep other cost measures low, such as size, weight, degree, and diameter, are important structures in areas such as VLSI design, distributed computing, and robotics. Proving the existence of spanners that simultaneously keep several cost measures low and efficiently computing such spanners for a given

*Department of Computer Science, University of Copenhagen, koolooz@diku.dk, http://www.diku.dk/~koolooz/
set of vertices is an active area of research \cite{2, 9}. For more on spanners, see surveys \cite{6, 12}, and the book by Narasimhan and Smid \cite{11}.

An interesting dual problem is the following: given a geometric graph, compute its stretch factor. A related problem which we consider in this paper is that of computing the maximum detour of a plane graph, which is defined like stretch factor except that the maximum is taken over all pairs of distinct points of the graph, not just the vertices.

If the graph is planar then its stretch factor can be computed in $\Theta(n^2)$ time, where $n$ is the number of vertices, by applying the APSP algorithm by Frederickson \cite{7}. This bound also holds for the problem of computing the maximum detour of a plane graph \cite{1}. It is an open problem whether subquadratic time algorithms exist.

For more special types of graphs, faster algorithms are known. If the graph is a path, its stretch factor and maximum detour can be computed in $O(n \log n)$ expected time and if the graph is a tree or a cycle, its stretch factor and maximum detour can be computed in $O(n \log^2 n)$ expected time \cite{1}. Using parametric search gives $O(n \text{polylog } n)$ worst-case time algorithms for these special types of graphs. Also, if the graph has bounded treewidth $k$, its stretch factor can be computed in $O(n \log^{k+1} n)$ expected time \cite{3}.

In this paper, we show how to compute the maximum detour of a plane graph with $n$ vertices in $O(n^{3/2} \log^3 n)$ expected time, thereby solving the open problem of whether a subquadratic time algorithm exists for this problem. We also show that if the graph has bounded treewidth, its maximum detour can be computed in $O(n \log^3 n)$ expected time.

The organization of the paper is as follows. In Section 2, we give various definitions and introduce some notation. In Section 3, we make use of the separator theorem by Lipton and Tarjan which enables us to apply the divide-and-conquer paradigm to the input graph. We define colourings of points of a face of the graph in Section 4, show some properties of these colourings and how to efficiently compute them. In Sections 5 and 6, we show how the colourings give an efficient way of computing the maximum detour between points on a face of the graph and this in turn gives an efficient algorithm for computing the maximum detour of the entire graph. In Section 7, we apply the same ideas to plane graphs with bounded treewidth. Finally, we make some concluding remarks in Section 8.
2 Definitions and Notation

Let $G = (V,E)$ be a plane graph where each edge is a line segment and let $P_G$ be the set of points of $G$ (vertices as well as interior points of edges). Given two points $p,q \in P_G$, we define $d_G(p,q)$ as the length of a shortest path in $G$ between $p$ and $q$, where the length of a path is measured as the sum of the Euclidean lengths of the (parts of) edges on this path. If there is no such path, we define $d_G(p,q) = \infty$. If $p \neq q$ then the detour $\delta_G(p,q)$ between $p$ and $q$ (in $G$) is defined as the ratio $d_G(p,q)/|pq|$. The maximum detour $\delta_G$ of $G$ is the maximum of this ratio over all pairs of distinct points of $P_G$.

Where appropriate, we will regard a plane graph as the set of points belonging to the graph. So for instance, if $G$ and $H$ are plane graphs then $G \cap H$ is the set of points belonging to both $G$ and $H$. If well-defined, we will regard the resulting point set as a graph.

For a graph $G$, we let $|G|$ denote its size, i.e. the number of vertices plus the number of edges in $G$. Given two subsets $P_1$ and $P_2$ of the set $P_G$ of points of $G$, we define

$$\delta_G(P_1, P_2) = \max_{p \in P_1, q \in P_2, p \neq q} \delta_G(p, q)$$

If $p$ is a point of $G$ and $P \subseteq P_G$, we write $\delta_G(p,P) = \delta_G(P,p)$ instead of $\delta_G(\{p\},P)$ and we write $\delta_G(P)$ as a shorthand for $\delta_G(P,P)$. We extend these definitions to subgraphs, edges, and vertices by regarding them as sets of points.

Given paths $P = p_1 \rightarrow p_2 \rightarrow \ldots \rightarrow p_r$ and $Q = q_1 \rightarrow q_2 \rightarrow \ldots \rightarrow q_s$, where $p_r = q_1$, we let $PQ$ denote the combined path $p_1 \rightarrow p_2 \rightarrow \ldots \rightarrow p_{r-1} \rightarrow q_1 \rightarrow q_2 \rightarrow \ldots \rightarrow q_s$. For a vertex $v$, we let $v \rightarrow P$ denote the path $v \rightarrow p_1 \rightarrow p_2 \rightarrow \ldots \rightarrow p_r$.

3 Separating the Problem

In all the following, let $G = (V,E)$ be an $n$-vertex plane graph in which edges are line segments. We seek to compute $\delta_G$ in $O(n^{3/2} \log^3 n)$ expected time. We will assume that $G$ is connected since otherwise, the problem is trivial.

To compute $\delta_G$, we apply the divide-and-conquer paradigm. In this section, we separate $G$ into two smaller graphs, $G_A$ and $G_B$, of roughly the same
size. We recursively compute $\delta_G(G_A)$ and $\delta_G(G_B)$ and in Section 5, we show how to efficiently obtain the maximum detour $\delta_G(G_A, G_B)$.

To separate our problem, we use the separator theorem of Lipton and Tarjan [10]. This gives us, in $O(n)$ time, a partition of $V$ into three sets, $A$, $B$, and $P$, such that

1. no edge joins a vertex in $A$ with a vertex in $B$,
2. neither $A$ nor $B$ contains more than $n/2$ vertices, and
3. $P$ contains no more than $\frac{\sqrt{n}}{1 - \sqrt{2/3}} \sqrt{n}$ vertices.

We refer to vertices of $P$ as portals. In all the following, we let $k$ denote the number of portals and let $p_1, \ldots, p_k$ denote the portals.

Having found this partition, we compute and store shortest path lengths from each portal to each vertex of $V$. This can be done in $O(n^{3/2} \log n)$ time using Dijkstra’s SSSP algorithm for each portal.

Let $G_A$ be the subgraph of $G$ induced by $A \cup P$ and let $G_B$ be the subgraph of $G$ induced by $B \cup P$. We construct $G_A$ and $G_B$ and recursively compute $\delta_G(G_A)$ and $\delta_G(G_B)$. Clearly,

$$\delta_G = \max\{\delta_G(G_A), \delta_G(G_B), \delta_G(G_A, G_B)\}.$$  

In the following, we deal with the problem of computing $\delta_G(G_A, G_B)$ and in Section 6, we show how to recursively compute $\delta_G(G_A)$ and $\delta_G(G_B)$.

We will need the following lemma which is a generalization of a result in [5] (we omit the proof since it is virtually identical to that in [5]).

**Lemma 1.** Maximum detour $\delta_G$ is achieved by a pair of co-visible points.

This result allows us to consider only detours between pairs of points of the same face of $G$. In all the following, let $f$ be a face of $G$, let $f_A = f \cap G_A$, and let $f_B = f \cap G_B$. We will show how to compute $\delta_G(f_A, f_B)$ in $O(|f|k \log^2 n)$ expected time. From this it will follow that $\delta_G(G_A, G_B)$ can be found in $O(n^{3/2} \log^2 n)$ expected time.

We will assume that $f$ is an internal face of $G$. The external face is dealt with in a similar way. We also assume that $f_A$ and $f_B$ do not share any edges since any edge $e$ shared by them must have portals as both endpoints and the detours from points in $e$ to all points in $G$ will be considered in the two
recursive calls that compute $\delta_G(G_A)$ and $\delta_G(G_B)$, respectively. Hence, we may disregard $e$ when computing $\delta_G(f_A, f_B)$.

Assume for now that $f$ is simple. The case where $f$ is non-simple is considered in Section 5.3.

4 Colouring Points

We now define colourings of points of $f$. As we shall see in Section 5, these colourings will prove helpful when computing $\delta_G(f_A, f_B)$.

Face $f$ is defined by a simple cycle $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow v_1$ such that the interior of $f$ is to the left as we walk in the direction specified by the vertices.

For a point $p \in f$, we let $d_f(p)$ denote the Euclidean length of the path from $v_1$ to $p$ that visits vertices in the order specified above. We define an order $<_p$ of the set of points of $f$ as follows. For two points $p$ and $q$ of $f$, $p <_p q$ if and only if $d_f(p) < d_f(q)$. Order $>_p q$ is defined in a similar way. In the following, we assume that the sum of lengths of all edges in $f$ and $d_f(v_i)$ for all $v_i \in f$ are precomputed.

By starting the walk in any other vertex $v_i$ of $f$, we can similarly define orders $<_i$ and $>_i$. It is easy to see that determining whether e.g. $p <_i q$ holds for any points $p, q \in f$ can be done in constant time using the above precomputed values. Where appropriate, we will regard the points of $f$ (or $f_A$ or $f_B$) occurring between two points w.r.t. order $<_i$ as an interval of points.

In the following, let $a$ be a vertex in $f_A$ and let $a' \in f \cap V$ be its successor w.r.t. $<_a$. We now consider associating with $a$ a colouring of points in $f_B$ using colours $c_1, \ldots, c_k$. A point $p \in f_B$ is given colour $c_i$ if portal $p_i$ is on a shortest path in $G$ from $a$ to $p$. In case of ties, pick the colour such that the corresponding portal has minimum distance to $a$ in $G$. In case of further ties, pick the colour with the smaller index. We let $c_a(p)$ denote the colour assigned to $p$.

We will show that colours occur in intervals as we walk around $f_B$ with each colour assigned to at most one interval. Furthermore, we will show that the order of these intervals is induced by an order of the portals which we define next.

Let $u_0$ and $u_1$ be distinct vertices of $G$ connected by an edge and consider a portal $p_i$. Choose edge $(u_1, u_2) \in E$ such that $u_2$ is on a shortest path from $u_1$ to $p_i$ and such that $u_0 \rightarrow u_1 \rightarrow u_2$ makes the sharpest possible left turn
at \( u_1 \) (if a left turn is not possible we regard the least possible right turn as a sharpest possible left turn and we regard a turn of angle \( \pi \) as a left turn of angle \( \pi \)).

Repeat this procedure by picking, for \( j = 3, \ldots, r \), an edge \((u_{j-1}, u_j)\) such that \( u_j \) is on a shortest path from \( u_{j-1} \) to \( p_i \) and such that \( u_{j-2} \rightarrow u_{j-1} \rightarrow u_j \) makes the sharpest possible left turn at \( u_{j-1} \); here, \( r \) is the smallest index such that \( u_j = p_i \). The resulting path \( u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_r \) is uniquely defined and is a shortest path from \( u_1 \) to \( p_i \). We denote it by \( \overrightarrow{P_i}(u_0, u_1) \). We define \( \overrightarrow{P'_i}(u_0, u_1) = u_0 \rightarrow \overrightarrow{P_i}(u_0, u_1) \). In the following, we will write \( \overrightarrow{P_i} \) resp. \( \overrightarrow{P_i}' \) as a shorthand for \( \overrightarrow{P_i}(a', a) \) resp. \( \overrightarrow{P_i}'(a', a) \), where \( a \) and \( a' \) are defined as above.

For two distinct portals \( p_i \) and \( p_j \) we write \( p_i \prec_a p_j \) if \( p_i \in \overrightarrow{P_j} \) or if \( \overrightarrow{P_i}' \) makes a sharpest left turn than \( \overrightarrow{P_j}' \) at some shared interior vertex.

**Lemma 2.** If paths \( \overrightarrow{P_i} \) and \( \overrightarrow{P_j} \) split at a vertex \( v \) they cannot meet after \( v \).

**Proof.** Suppose the lemma does not hold. Let \( w \) be a vertex where \( \overrightarrow{P_i} \) and \( \overrightarrow{P_j} \) meet after \( v \). Assume w.l.o.g. that \( \overrightarrow{P_i} \) makes a sharper left turn at \( v \) than \( \overrightarrow{P_j} \). Replacing the subpath of \( \overrightarrow{P_j} \) from \( v \) to \( w \) by the subpath of \( \overrightarrow{P_i} \) from \( v \) to \( w \) gives a shortest path from \( a \) to \( p_j \) but this contradicts the assumption that \( \overrightarrow{P_j} \) makes the sharpest possible left turn at \( v \). \( \square \)

**Lemma 3.** The relation \( \prec_a \) above is a strict total order of the portals.

**Proof.** Let \( p_{i_1}, p_{i_2}, \) and \( p_{i_3} \) be three distinct portals. Clearly, \( p_{i_1} \prec_a p_{i_2} \) or \( p_{i_2} \prec_a p_{i_1} \), showing that the relation \( \prec_a \) is total.

We need to show that \( p_{i_2} \prec_a p_{i_2} \) and \( p_{i_2} \prec_a p_{i_1} \) cannot both hold and we need to show transitivity: if \( p_{i_1} \prec_a p_{i_2} \) and \( p_{i_2} \prec_a p_{i_3} \) then \( p_{i_1} \prec_a p_{i_3} \).

To show the first part, suppose for the sake of contradiction that \( p_{i_1} \prec_a p_{i_2} \) and \( p_{i_2} \prec_a p_{i_1} \). Then we cannot have \( p_{i_1} \in \overrightarrow{P_{i_2}} \) and we cannot have \( p_{i_2} \in \overrightarrow{P_{i_1}} \). Consider the first vertex \( v \) at which paths \( \overrightarrow{P_{i_1}} \) and \( \overrightarrow{P_{i_2}} \) split. Then they cannot meet after \( v \) by Lemma 2. But this implies that one of the two paths \( \overrightarrow{P_{i_1}} \) and \( \overrightarrow{P_{i_2}}' \) cannot make a sharpest left turn than the other path at any shared interior vertex, a contradiction.

To show transitivity, suppose \( p_{i_1} \prec_a p_{i_2} \) and \( p_{i_2} \prec_a p_{i_3} \). We need to show that either \( p_{i_1} \in \overrightarrow{P_{i_3}} \) or that \( \overrightarrow{P_{i_1}}' \) makes a sharpest left turn than \( \overrightarrow{P_{i_3}}' \) at some interior vertex of \( \overrightarrow{P_{i_3}}' \).
In particular, by symmetry, it is enough to show that if $p \in \overline{P_{i_2}}$, then $p_{i_1} \in \overline{P_{i_3}}$ (Figure 1(i)). If $p_{i_2} \notin \overline{P_{i_3}}$ then $\overline{P_{i_2}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at some interior vertex $v$ of $\overline{P_{i_3}}'$. If $v \in \overline{P_{i_1}} \setminus \{p_{i_1}\}$ (Figure 1(ii)) then $\overline{P_{i_1}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at $v$. And if $v \notin \overline{P_{i_1}} \setminus \{p_{i_1}\}$ (Figure 1(iii)) then $p_{i_1} \in \overline{P_{i_3}}$. In all cases, $p_{i_1} \prec_a p_{i_3}$.

Now, assume that $p_{i_1} \notin \overline{P_{i_2}}$. Then $\overline{P_{i_1}}'$ makes a sharper left turn than $\overline{P_{i_2}}'$ at some interior vertex $v$ of $\overline{P_{i_2}}'$. If $p_{i_2} \notin \overline{P_{i_3}}$ (Figure 1(iv)) then $\overline{P_{i_1}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at $v$. If $p_{i_2} \notin \overline{P_{i_3}}$ then let $w$ be a vertex such that $\overline{P_{i_2}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at $w$. If $w \not\in \overline{P_{i_1}}$ (Figure 1(v)) then $\overline{P_{i_1}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at $w$. And if $w \not\in \overline{P_{i_1}}$ (Figure 1(vi)) then $\overline{P_{i_1}}'$ makes a sharper left turn than $\overline{P_{i_3}}'$ at $v$. So again, $p_{i_1} \prec_a p_{i_3}$. □

We now show the relation between the order $\prec_a$ of portals and the order $\prec_f$ of vertices in $f_B$.

**Theorem 1.** Let $p$ and $q$ be two distinct points of $f_B$ and assume that $c_a(p) = c_i$ and $c_a(q) = c_j$, $i \neq j$. Then $p_i \prec_f p_j$ if and only if $p \prec_a q$.

**Proof.** By symmetry, it is enough to show that if $p_i \prec_a p_j$ then $p \prec_f q$.

So assume that $p_i \prec_a p_j$. Since $c_a(q) = c_j$, we have $p_i \not\in \overline{P_j}$. It follows that there is a vertex $v$ at which $\overline{P_i}$ and $\overline{P_j}$ split and $\overline{P_i}$ makes a sharper left turn at $v$ than $\overline{P_j}'$. By Lemma 2, the two paths do not meet again after $v$. In particular, $\overline{P_i}$ does not cross $\overline{P_j}$, see Figure 2.

Let $P_p$ be a shortest path from $p_i$ to $p$. Then $P_p$ cannot intersect $\overline{P_j}$ since then $p_i$ and $p_j$ would both be on a shortest path from $a$ to $q$ with $d_G(a, p_i) < d_G(a, p_j)$, contradicting the assumption that $c_a(q) = c_j$.

Figure 1: The six cases considered in Lemma 3.
Let $P_q$ be a shortest path from $p_j$ to $q$. Then $\overrightarrow{P_i}$ cannot intersect $P_q$ since otherwise, $p_i$ and $p_j$ would both be on a shortest path from $a$ to $p$ with $d_G(a, p_j) < d_G(a, p_i)$, contradicting the assumption that $c_a(p) = c_i$.

Furthermore, $P_q$ cannot intersect $P_p$. For assume it did. Then there would be a shortest path from $a$ to $p$ through $p_j$ and a shortest path from $a$ to $q$ through $p_i$. If $d_G(a, p_i) < d_G(a, p_j)$ then $c_a(q) \neq c_j$ and if $d_G(a, p_j) < d_G(a, p_i)$ then $c_a(p) \neq c_i$. Hence, $d_G(a, p_i) = d_G(a, p_j)$. But then $i < j$ would imply $c_a(q) \neq c_j$ and $i > j$ would imply $c_a(p) \neq c_i$. This contradicts the colours assigned to $p$ and $q$.

It follows from the above that paths $\overrightarrow{P_i}P_p$ and $\overrightarrow{P_j}P_q$ do not intersect except in the vertices they share until reaching $v$, see Figure 2. This implies that $p <_a q$, showing the theorem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The possible situations in the proof of Theorem 1 when $p_i <^l_a p_j$.}
\end{figure}

Corollary 1. Interval $f_B$ can be split up into $O(k)$ sub-intervals such that points in the same sub-interval are assigned the same colour w.r.t. vertex $a \in f_A$.

If the sub-intervals of Corollary 1 are picked such that they have maximal size, we refer to them as colour intervals of $a$.

We now show how the colour intervals of $a$ can be computed efficiently, after some preprocessing.

Lemma 4. The order of colour intervals of $a$ can be computed in $O(k \log^2 n)$ time assuming $O(kn \log n)$ time for preprocessing. The preprocessing step is independent of $a$ and $f$.

Proof. We will prove the lemma by presenting a data structure that, given distinct portals $p_i$ and $p_j$, determines whether $p_i <^l_a p_j$ and does so for any
Figure 3: Example with $\pi$-pointers from edge $e$ to edges $e_0 = \pi_0(e, i)$, $e_1 = \pi_1(e, i)$, $e_2 = \pi_2(e, i)$, and $e_3 = \pi_3(e, i)$ on $\overrightarrow{P_i}(e)$. Note that $e_1 = \pi_0(\pi_0(e, i), i)$, $e_2 = \pi_1(\pi_1(e, i), i)$, and $e_3 = \pi_2(\pi_2(e, i), i)$. Here, $j_e = 3$. 

We will show how to construct this data structure in $O(kn \log n)$ time such that it can determine whether $p_i \preceq_a p_j$ in $O(\log n)$ time. This will allow us to sort the portals according to $\preceq_a$ in time $O(k \log^2 n)$ time using a sorting algorithm like merge or heap sort since such an algorithm performs $O(k \log k)$ comparisons. This result together with Theorem 1 will show the lemma.

We first show how to construct the data structure. In the following, let $E'$ be the set of directed edges obtained by regarding each edge of $E$ as two oppositely directed edges.

With each edge $e = (u, v) \in E'$ and each portal $p_i$, we associate pointers $\pi_j(e, i)$, $j = 0, \ldots, j_e$. Pointer $\pi_j(e, i)$ points to the edge $e_j$ of $\overrightarrow{P_i}(e)$ such that the number of edges between $e$ and $e_j$ in $\overrightarrow{P_i}(e)$ is $2^j - 1$, see Figure 3. Here, $j_e$ is the largest $j$ such that $e_j$ exists. Note that $j_e = O(\log n)$ so the total number of pointers over all edges $e$ and all portals $p_i$ is $O(kn \log n)$.

Suppose we are given these pointers. Then for any edge $e$ and any portals $p_i$ and $p_j$, binary search allows us to determine, in $O(\log n)$ time, the vertex at which $\overrightarrow{P_i}(e)$ and $\overrightarrow{P_j}(e)$ split. If they do not split, binary search will also detect this and determine whether $p_i \in \overrightarrow{P_j}(e)$ or $p_j \in \overrightarrow{P_i}(e)$. From this it follows that we can determine whether $p_i \preceq_a p_j$ in $O(\log n)$ time, given these pointers.

We now show how to construct the pointers in $O(kn \log n)$ time. More
specifically, given a portal \( p_i \), we show how to compute pointers \( \pi_j(e, i) \), \( e \in E, j = 1, \ldots, j_e \), in a total of \( O(n \log n) \) time.

For each edge \((u, v)\) of \( E'\), we colour it white if a shortest path from \( u \) to \( p_i \) goes through \( v \), that is, if \( d_G(u, p_i) = |uv| + d_G(v, p_i) \). Otherwise, we colour it black.

Now, for each vertex \( v \) of \( G \), we consider the edges of \( E' \) starting or ending in \( v \) in counter-clockwise order. For each edge \((u, v)\) ending in \( v \), we add a pointer to the previous (in the counter-clockwise order) white edge \((v, w)\) that starts in \( v \). Note that \((v, w)\) is the edge satisfying that \( u \to v \to w \) makes the sharpest possible left turn such that \((v, w)\) is on a shortest path in \( G \) from \( v \) to \( p_i \).

Next, we consider the edges of \( E' \) in some order \( e_1, \ldots, e_{|E'|} \). For edge \( e_1 = (u_1, v_1) \), we compute \( \overrightarrow{P_i}(e_1) \) by starting in \( v_1 \) and then computing subsequent vertices using the pointers just added. We colour \( e_1 \) red and as we visit the edges of \( \overrightarrow{P_i}(e_1) \), we colour them red as well.

We then compute pointers \( \pi_j(e, i) \) for each edge \( e \in \overrightarrow{P_i}(e_1) \) and for each \( j = 0, \ldots, j_e \). Below, we show how to do this efficiently.

For edge \( e_j, j > 1 \), we do exactly the same as for \( e_1 \) except that we stop when reaching a red edge. All visited edges including \( e_j \) are coloured red and \( \pi \)-pointers are computed for each of these edges.

The above algorithm is correct since it computes \( \pi \)-pointers for all edges. Its running time, excluding the time to compute \( \pi \)-pointers, is \( O(n \log n) \). To see this, note that sorting edges counter-clockwise for each vertex takes a total of \( O(n \log n) \) time. Colouring edges black and white takes \( O(n) \) time. Finally, visiting (parts of) paths \( \overrightarrow{P_i}(e_j) \) takes time proportional to the number of edges coloured red which is \( O(n) \).

What remains is to show how to compute \( \pi \)-pointers in \( O(n \log n) \) time. We observe that \( \pi_j(e, i), j = 0, \ldots, j_e \), can be computed in a total of \( O(\log n) \) time if \( \pi \)-pointers of all edges of \( \overrightarrow{P_i}(e) \) have been computed. This follows easily from repeated applications of the identity \( \pi_j(e, i) = \pi_{j-1}(\pi_{j-1}(e, i), i), j > 0 \) (see Figure 3). Hence, by computing \( \pi \)-pointers in the opposite order of the order in which edges are coloured red, we obtain all \( \pi \)-pointers in \( O(n \log n) \) time.

\( \square \)

**Theorem 2.** Endpoints of colour intervals of \( a \) can be computed in \( O(k \log^2 n) \) time assuming \( O(kn \log n) \) preprocessing time. The preprocessing step is independent of \( a \) and \( f \).
Proof. We start by computing the order of colour intervals of \( a \). By Lemma 4, this can be done in \( O(k \log^2 n) \) time with \( O(kn \log n) \) preprocessing time. Let \( \pi \) be the permutation of \( \{1, \ldots, k\} \) such that \( p_{\pi(1)} \prec_f \cdots \prec_f p_{\pi(k)} \).

We then compute the colour of the first point \( b_{\min} \) and the last point \( b_{\max} \) of \( f_B \) w.r.t. the order \( \prec_f \). This can be done in time proportional to the number \( k \) of colours since SSSP lengths for each portal have been precomputed.

If \( c_a(b_{\min}) = c_a(b_{\max}) \) then by Theorem 1, all points between \( b_{\min} \) and \( b_{\max} \) have this colour. In this case, the algorithm associates this colour with the sub-interval between the two vertices and returns (the sub-interval is not stored explicitly, only its end vertices \( b_{\min} \) and \( b_{\max} \)).

Otherwise, a vertex \( b \in f_B \) is picked, such that the number of edges in \( f_B \) before resp. after \( b \) w.r.t. the order \( \prec_f \) is (approximately) the same, and its colour \( c_a(b) \) is computed. Let \( i \) be the index such that \( c_{\pi(i)} = c_a(b) \). The algorithm calls itself recursively on vertices between \( b_{\min} \) and \( b \) with colours \( c_{\pi(1)}, \ldots, c_{\pi(i)} \). And it calls itself recursively on vertices between \( b \) and \( b_{\max} \) with colours \( c_{\pi(i)}, \ldots, c_{\pi(k)} \).

The recursion stops when \( b_{\min} \) and \( b_{\max} \) are the endpoints of a single edge \( e \) of \( f_B \). If \( c_a(b_{\min}) = c_a(b_{\max}) \), the algorithm associates this colour with \( e \) and returns. Otherwise, we need the following simple observation: there is a point \( p \) on \( e \) such that all points on \( b_{\min}p \) have colour \( c_a(b_{\min}) \) and all points on \( pb_{\max} \) have colour \( c_a(b_{\max}) \). Furthermore, \( p \) can be computed in \( O(1) \) time, given these two colours. The algorithm associates the two colours with their respective segments of \( e \) and returns.

When the algorithm terminates, each colour \( c_i \) is associated with \( O(\log n) \) sub-intervals and their union defines the colour interval of \( a \) with colour \( c_i \). Finding the colour intervals of \( a \) from these \( O(k \log n) \) sub-intervals takes \( O(k \log n) \) time. What remains is to show that the algorithm above has \( O(k \log^2 n) \) running time.

Let \( T(m, k) \) be a function expressing the time for the above algorithm where \( m \) is the number of edges and \( k \) is the number of colours. If we assume that vertices of \( f_B \) are stored in an array then the point \( b \) that splits points of \( f_B \) into two equal halves can be found in \( O(1) \) time. Thus, there is a constant \( c' > 0 \) such that the algorithm uses at most \( c'k \) time steps excluding time spent in recursive calls. There is also a constant \( c'' \) such that \( T(m, k) \leq c''k \) when \( m \leq 2 \). Let \( c = \max\{c', c''\} \). Then (ignoring floors and ceilings)

\[
T(m, k) \leq ck + T(m/2, k_1) + T(m/2, k_2),
\]

where \( m > 2 \) and \( k_1, k_2 \in \{1, \ldots, k\} \), \( k_1 + k_2 = k + 1 \). Let \( \tilde{T}(m, k) \) be the
running time $T(m, k)$ minus a value of $c \log n$ charged to each split vertex $b$
encountered in the current and in recursive calls. We claim that $\tilde{T}(m, k) \leq
ck \log m$.

The proof is by induction on $m \geq 2$. When $m = 2$, we spend at most
$c'k \leq ck$ time. Now, assume that $m > 2$. The induction hypothesis gives

$$\tilde{T}(m, k) \leq ck + \tilde{T}(m/2, k_1) + \tilde{T}(m/2, k_2) - c \log n$$

as requested.

It follows that $T(m, k) \leq ck \log m + xc \log n$, where $x$ is the total number
of split vertices. This number is proportional to the number of sub-intervals
returned by the algorithm which is $O(k \log m)$. This shows the theorem. \[\square\]

Note that a colour interval $I$ need not be closed, i.e. one or both of the
endpoints need not belong to $I$.

We conclude this section with the following simple result which will prove
useful when we compute detours between points that may be interior points
of edges.

**Lemma 5.** Let $p$ a point of edge $e = (u, v)$ of $G$ and let $q$ be a point in $G$.
Suppose that $p_i$ is on a shortest path from $u$ to $q$ and that $p_j$ is on a shortest
path from $v$ to $q$. Then either $p_i$ or $p_j$ is on a shortest path from $p$ to $q$.

**Proof.** A shortest path from $p$ to $q$ goes through either $u$ or $v$. \[\square\]

## 5 The Detour of Points in a Face

In this section, we show how to compute $\delta_G(f_A, f_B)$ in $O(|f|k \log^2 n)$ expected
time.

We start by computing, for each edge $e = (u, v) \in f_A$, $O(k)$ colour
intervals of $u$ and of $v$ using $O(k \log^2 n)$ time (with $O(kn \log n)$ preprocessing)
and take the union of the endpoints of these colour intervals. This gives $O(k)$
smaller sub-intervals which we associate with $e$. The total running time for
this over all edges is $O(|f|k \log^2 n)$.

Now, let $P$ be one of the sub-intervals associated with edge $e = (u, v)$.
Then there are $i, j \in \{1, \ldots, k\}$ such that $c_u(p) = c_i$ and $c_v(p) = c_j$ for all
Let \( p \in P \). Hence, for any point \( q \in P \), \( p_i \) is on a shortest path from \( u \) to \( q \) and \( p_j \) is on a shortest path from \( v \) to \( q \). Lemma 5 implies that for any point \( p \in e \) and any point \( q \in P \), either \( p_i \) or \( p_j \) is on a shortest path from \( p \) to \( q \).

We refer to \( P \) as a type 1-interval (of \( e \)) if \( c_i \neq c_j \) and a type 2-interval (of \( e \)) if \( c_i = c_j \).

For any edge \( e \) of \( f_A \) and any type \( i \)-interval \( P \) of \( e \), \( i = 1, 2 \), we may assume that \( P \) is a closed interval having endpoints in vertices of \( f_B \). For otherwise, we could compute the maximum detour between \( e \) and the first resp. last edge \( e' \) of \( P \) (all other edges of \( P \) have endpoints in vertices of \( f_B \)). Computing \( \delta_G(e, e') \) is a constant-size problem (when SSSP lengths for each portal have been precomputed) since we know that for each point \( p \in e \) and each point \( p' \in e \), there is a shortest path from \( p \) to \( p' \) through either of two portals \( p_i \) and \( p_j \). Thus, it takes \( O(1) \) time to compute \( \delta_G(e, e') \) (see also [1]). Over all \( e \) and \( P \), this amounts to \( O(|f|k) \) time.

The value \( \delta_G(f_A, f_B) \) is computed in two phases. In phase \( i \), the maximum detour between points in edges of \( f_A \) and points in associated type \( i \)-intervals are computed, \( i = 1, 2 \).

### 5.1 Phase 1

We will show that phase 1 takes \( O(|f|k) \) time when shortest path lengths from portals and colour intervals have been computed.

The algorithm for this phase is straightforward. For each edge \( e \) of \( f_A \) it considers all edges \( e' \) of each type 1-interval of \( e \) and computes \( \delta_G(e, e') \) in constant time.

To show that phase 1 takes \( O(|f|k|) \) time, we need to show that the number of edge pairs \((e, e')\) is \( O(|f|k|) \). To do this, we introduce a so called dual colouring of vertices of \( f_A \) for each vertex of \( f_B \). Let \( b \) be a vertex of \( f_B \). Then vertex \( a \in f_A \) is given dual colour \( c_b(a) \), defined as the colour \( c_b(b) \).

Assigning dual colours to all vertices of \( f_A \) partitions this set into maximal sub-intervals with vertices in each sub-interval having the same dual colour. We refer to these sub-intervals as the dual colour intervals of \( b \). These dual colour intervals will help us bound the number of edge pairs \((e, e')\). First, we need two lemmas.

**Lemma 6.** Let \( b \) be a vertex of \( f_B \) and let \( a, a_1, a_2 \) be vertices of \( f_A \) such that \( a_1 <_b a <_b a_2 \), \( c_b(a_1) = c_b(a_2) = c_i \), and \( c_b(a) = c_j \), \( i \neq j \). Then for any vertex \( a' \) of \( f_A \), \( c_b(a') \neq c_j \) if either \( a' <_b a_1 \) or \( a' >_b a_2 \).
Figure 4: The situations considered in the proof of Lemma 6.

Proof. Let \( a' \) be a vertex of \( f_A \) such that either \( a' <_b a_1 \) or \( a' >_b a_2 \). Portal \( p_i \) is on some shortest path \( P_1 \) from \( a_1 \) to \( b \). Pick \( P_1 \) such that the subpath \( P'_1 \) from \( a_1 \) to \( p_i \) makes sharpest possible left turns as described in the previous section. In a similar way, we define \( P_2 \) and \( P'_2 \) for \( a_2 \). Note that any path from \( a \) to \( b \) must intersect either \( P'_1 \) or \( P'_2 \), see Figure 4.

Portal \( p_j \) is on a shortest path \( P \) from \( a \) to \( b \). Pick \( P \) such that the subpath \( P' \) from \( a \) to \( p_j \) makes sharpest possible left turns. Path \( P' \) cannot cross \( P_1 \) for otherwise \( c(a)(b) \neq c_j \), and \( p_j \notin P'_1 \) for otherwise \( c_{a_i}(b) \neq c_i \). Similarly, \( P' \) cannot cross \( P_2 \) for otherwise \( c_{a_2}(b) \neq c_i \), and \( p_j \notin P'_2 \) for otherwise \( c_{a_2}(b) \neq c_i \). Furthermore, \( p_j \) cannot belong to the subpath shared by \( P_1 \) and \( P_2 \) from \( p_i \) to \( b \) since then \( c_{a}(b) \neq c_j \).

It follows that any shortest path from \( a' \) to \( p_j \) crosses either \( P'_1 \) or \( P'_2 \). This implies that \( c_b(a') = c_{a'}(b) \neq c_j \), as requested.

Lemma 7. The number of dual colour intervals of a vertex \( b \in f_B \) is \( O(k) \).

Proof. Let \( N(k) \) be the maximum number of dual colour intervals of \( b \) when the number of distinct colours in these intervals is exactly \( k \). We will show that \( N(k) \leq 2k - 1 \). The proof is by induction on \( k \geq 1 \). If \( k = 1 \) there is only one dual colour interval and we have \( N(k) = 1 = 2k - 1 \).

Now, suppose that \( k > 1 \) and that \( N(k') \leq 2k' - 1 \) for all \( k' \) less than \( k \). Let \( c_i \) be the colour of the first dual colour interval w.r.t. the order \( <_b \). By Lemma 6, there is a finite number \( r \) of dual colour intervals with colour \( c_i \) (in fact at most \( k \)). Let \( I_1, \ldots, I_s \) be the intervals between each consecutive pair of these dual colour intervals and let \( k_1, \ldots, k_s \) be the number of colours in each of them. Note that \( s \geq r - 1 \). Also note that by the choice of \( c_i \) and by Lemma 6, two points in two different \( I \)-intervals cannot have the
same colour since for at least one of the two intervals, the colour of the dual colour interval preceding and succeeding it is \( c_i \). From this and from the fact that the \( I \)-intervals do not contain colour \( c_i \),
\[
\sum_{j=1}^{s} k_j = k - 1.
\]
Applying the induction hypothesis, this gives
\[
N(k) \leq r + \sum_{j=1}^{s} N(k_j) \leq r + \sum_{j=1}^{s} 2k_j - 1 = r - s + 2(k - 1) \leq 1 + 2(k - 1) = 2k - 1.
\]

We are now ready to bound the running time for the phase 1 algorithm.

**Theorem 3.** The algorithm for phase 1 has \( O(|f|k) \) running time.

**Proof.** Consider an edge \( e = (u, v) \) of \( f_A \) and an edge \( e' = (u', v') \) of \( f_B \) such that \( e' \) is an edge of a type 1-interval of \( e \). Let \( c_i = c_u(u') = c_u(v') \) and let \( c_j = c_v(u') = c_v(v') \). Since \( u' \) is a vertex of \( f_B \), dual colours \( c_u'(u) = c_i \) and \( c_u'(v') = c_j \) are well-defined and since \( i \neq j \) by assumption, these two dual colours are distinct, implying that two dual colour intervals of \( b \) meet at \( e \).

It follows by the above and by Lemma 7 that for each \( e' \), the phase 1 algorithm picks \( O(k) \) edges \( e \). Thus, the total number of edge pairs considered by the algorithm is \( O(|f|k) \). By our earlier discussion, this suffices to show the theorem.

### 5.2 Phase 2

We now consider the problem of computing the maximum detour between points of edges of \( f_A \) and points of type 2-intervals.

For any pair \((e, P)\), where \( e \) is an edge of \( f_A \) and \( P \) is a type 2-interval of \( e \), there is a portal \( p_i \) such that for any point \( p \in e \) and any point \( q \in P \), \( p_i \) is on a shortest path from \( p \) to \( q \). In the following, we consider all such pairs for a fixed \( p_i \). We will show that computing the maximum detour \( \delta_i \) over all these pairs can be done in \( O(|f| \log^2 |f|) \) expected time. From this, it will follow that phase 2 takes \( O(|f|k \log^2 |f|) \) expected time.

Before showing how to compute \( \delta_i \), we need the idea of a canonical decomposition of \( f_B \), defined next.
5.2.1 Canonical Decomposition

Define $b_1, \ldots, b_m$ as the interval of vertices of $f_B$ ordered according to $<_a^f$ for some arbitrary vertex $a \in f_A$. Consider splitting this interval at vertex $b_j = b_{\lceil m/2 \rceil}$ and repeat this process recursively on the two sub-intervals, stopping when an interval containing only two vertices is reached. This gives us $O(m) = O(|f|)$ intervals of total size $O(|f| \log |f|)$ which we refer to as canonical intervals. The subgraphs of $f_B$ induced by these canonical intervals are referred to as canonical subgraphs. The set of these subgraphs, which we denote by $\mathcal{C}$, can be found in $O(|f| \log |f|)$ time.

Every sub-interval of $b_1, \ldots, b_m$ can be decomposed into $O(\log |f|)$ canonical intervals in $O(\log |f|)$ time. This is easily seen by applying a greedy algorithm that picks canonical intervals as large as possible. We refer to such a decomposition as a canonical decomposition.

Let $e$ be an edge of $f_A$. By the assumption earlier that type 2-intervals end in vertices and by Theorem 1, the union of all type 2-intervals of $e$ of colour $c_i$ is exactly the set of points of $f_B$ between two vertices of $b_1, \ldots, b_m$. We compute a canonical decomposition of the sub-interval between these two vertices and add a pointer to $e$ from each canonical subgraph corresponding to canonical intervals in this decomposition. This is done for all $e$ in $f_A$.

When finished we have a total of $O(|f| \log |f|)$ pointers from canonical subgraphs in $\mathcal{C}$ to edges of $f_A$. Note that some canonical subgraphs may contain pointers to several edges. The total time spent on constructing $\mathcal{C}$ and on finding pointers is $O(|f| \log |f|)$.

Observe that $\delta_i$ is the maximum of $\delta_{G}(e, C)$ over all pairs consisting of an edge $e \in f_A$ and a canonical subgraph $C \in \mathcal{C}$ with a pointer to $e$. We now show how to compute this maximum in $O(|f| \log^2 |f|)$ expected time.

5.2.2 Sweep-plane Algorithm

To efficiently compute the maximum over pairs of edges and canonical subgraphs, we will use the idea of lifting and lowering points followed by a sweep-plane algorithm as described in [8]. In order to do this, we consider the following decision problem below: given $\delta \in \mathbb{R}$, is $\delta_i \geq \delta$? If we can answer this quickly we can compute $\delta_i$ in low expected time using a randomized algorithm by Chan [4] as described in [8].

For each canonical subgraph $C \in \mathcal{C}$, we lift each point $p \in C$ to height $d_G(p_i, p)$. And for each edge $e \in f_A$, we lower each point $p \in e$ to height $d_G(p, p_e)$. Then we sweep a horizontal line segment across the plane and keep track of the maximum height encountered. As we sweep, we can update the maximum height in $O(\log |f|)$ time per event. The total time spent is $O(|f| \log |f|)$.

Note that the sweep must be performed only once, and the maximum height can be computed in $O(|f| \log |f|)$ time by computing the maximum of the heights at each event. The total time spent on finding the maximum is $O(|f| \log^2 |f|)$ expected time.
\(-d_G(p, p_i)\). Since we have precomputed SSSP lengths for portal \(p_i\), this lifting/lowering can be done in \(O(|f| \log |f|)\) time since the total size of all canonical subgraphs and edges is \(O(|f| \log |f|)\).

Let \(e\) be a vertex in \(f_A\) and let \(C\) be a canonical subgraph with a pointer to \(e\). Then it is clear that the height difference between a point \(p \in e\) and a point \(q \in C\) equals \(d_G(p, q)\).

For each lifted and lowered point \(p\), we associate a cone extending downwards from \(p\) and spanning an angle of \(\alpha = 2 \arctan(1/\delta)\). Then as shown in [8], \(\delta_i \geq \delta\) if and only if a cone of a lowered point of some edge is contained in a cone of a lifted point of some canonical subgraph with a pointer to that edge.

Now, we sweep a plane over the cones. The sweep-plane is parallel to the \(x\)-axis and forms an angle of \((\pi - \alpha)/2\) with the \(xy\)-plane. During the sweep, we maintain, for each canonical subgraph \(C\), the intersection between the sweep-plane and the upper envelope of lifted points of \(C\) together with lowered points in edges that \(C\) points to. If it is detected that a cone of a lowered point is contained in a cone of a lifted point, the algorithm reports that \(\delta_i \geq \delta\) and if no such event occurs, the algorithm reports that \(\delta_i < \delta\). This solves our decision problem.

It follows from the results of [8] that maintaining intersections between the sweep-plane and upper envelopes takes a total of \(O(|f| \log^2 |f|)\) time since the number of sweep-plane event points is \(O(|f| \log |f|)\) and each event point takes \(O(\log |f|)\) time to handle. However, this is under the assumption that no cone of a lifted point is contained in the interior of another cone of a lifted point and that no cone of a lowered point is contained in the interior of another cone of a lowered point (see [8] for details).

Since all lifted points belong to \(G_B\), we may satisfy this assumption for cones of lifted points by solving our decision problem using only values \(\delta \geq \delta_G(G_B)\) (a similar trick is used in [8] for trees). For suppose that \(\delta \geq \delta_G(G_B)\) and consider two points \(p\) and \(p'\) belonging to the same canonical subgraph.

Let \(h\) be the height of \(p\) and let \(h'\) be the height of \(p'\). Assume w.l.o.g. that \(h \geq h'\). Then
\[
    h - h' = d_G(p, p_i) - d_G(p', p_i).
\]

We know that \(d_G(p, p') \leq \delta_G(G_B) |pp'| \leq \delta |pp'|\). This gives
\[
    \frac{h - h'}{|pp'|} = \frac{d_G(p, p_i) - d_G(p', p_i)}{|pp'|} \leq \frac{d_G(p, p')}{|pp'|} \leq \delta.
\]

17
Figure 5: (a): The walk of \( f \) when \( f \) is non-simple. (b): Theorem 1 holds also when \( f \) is non-simple.

showing that the cone associated with \( p' \) cannot belong to the interior of the cone associated with \( p \). A similar argument shows that with \( \delta \geq \delta_G(G_A) \), no cone of a lowered point is contained in the interior of another cone of a lowered point.

So by recursively computing \( \delta_G(G_B) \) and \( \delta_G(G_A) \) before computing \( \delta_G(G_A, G_B) \) it follows that the above decision problem can be solved in \( O(|f| \log |f|) \) time and Chan’s algorithm gives us the following result.

**Theorem 4.** The algorithm for phase 2 has \( O(|f|k \log^2 |f|) \) expected running time.

What remains in order to compute \( \delta_G(G_A, G_B) \) is to handle the case where \( f \) is non-simple. This is the topic of the next section.

### 5.3 Non-simple Faces

Suppose \( f \) is non-simple. The walk of \( f \) defined in the beginning of Section 4 still applies if we inflate the edges and allowing an edge to be visited twice, see Figure 5(a). Then running through the arguments of the preceding sections, it can be seen that all results still hold (as an example, compare Figure 5(b) with Figure 2).

We have now obtained the following result.

**Theorem 5.** \( \delta_G(G_A, G_B) \) can be computed in \( O(n^{3/2} \log^2 n) \) expected time.
6 Dealing with Recursive Calls

In the preceding sections, we have shown how to compute \( \delta_G(G_A, G_B) \) in \( O(n^{3/2} \log^2 n) \) time. In this section, we present an algorithm that recursively computes \( \delta_G(G_A) \) (computing \( \delta_G(G_B) \) is dealt with in a similar way) and we will analyze its running time. From this analysis it will follow that \( \delta_G \) can be computed in \( O(n^{3/2} \log^3 n) \) expected time.

Let \( n_A = |A| \) be the number of vertices of \( A \). We start by applying the separator theorem to \( G_A \) which partitions \( A \) into three sets, \( A_1, A_2, \) and \( C_A \) such that no edge joins a vertex in \( A_1 \) with a vertex in \( A_2 \), neither \( A_1 \) nor \( A_2 \) contains more than \( n_A/2 \) vertices, and \( C_A \) contains no more than \( \frac{2n}{\sqrt{4/3}} - \sqrt{n_A} \) vertices. Let \( G_{A_1} \) be the subgraph of \( G_A \) induced by \( A_1 \cup C_A \) and let \( G_{A_2} \) be the subgraph of \( G_A \) induced by \( A_2 \cup C_A \). We have

\[
\delta_G(G_A) = \max \{ \delta_G(G_{A_1}), \delta_G(G_{A_2}), \delta_G(G_{A_1}, G_{A_2}) \}.
\]

In the following, we will present an algorithm for computing \( \delta_G(G_{A_1}, G_{A_2}) \). From this it will easily follow how to handle subgraphs in any recursion level.

Let \( P \) be the set of portals for \( G \) and let \( P_A = P \cup C_A \). Given a vertex \( a_1 \in A_1 \) and a vertex \( a_2 \in A_2 \), any path in \( G \) from \( a_1 \) to \( a_2 \) must contain at least one vertex of \( P_A \). Thus, by defining \( P_A \) as the set of portals for \( G_{A_i} \), most of the results we presented in order to compute \( \delta_G(G_A, G_B) \) now also apply to the problem of computing \( \delta_G(G_{A_1}, G_{A_2}) \). However, three problems need to be dealt with.

The first problem is that \( G_A \) may be disconnected and it causes problems when defining cyclic orderings of faces. We deal with this as follows. Let \( f \) be a face of \( G \) such that \( f \cap A \neq \emptyset \). Then as the cyclic ordering of the vertices of \( f \cap A \), we use that which is induced by the cyclic ordering of vertices of \( f \). Essentially, this defines faces of \( G_A \) where “holes” are allowed.

The second problem is that of computing SSSP distances from each portal of \( P_A \) to all vertices in \( G_A \). The running time for this should depend on the size of \( G_A \), not the size of \( G \). To obtain this, note that we only need to compute SSSP distances from portals of \( C_A \) since we have computed distances from each portal in \( P \) to all vertices of \( G \) and in particular to vertices of \( G_A \).

So consider a portal \( p_i \in C_A \). We compute SSSP distances from \( p_i \) to vertices in \( G_A \) using Dijkstra’s algorithm but with a different initialization. We are already given distances from \( p_i \) to portals in \( P \) so we add these portals to the priority queue and associate a shortest path distance from \( p_i \) to each of
them. We also add \( p_i \) to the priority queue and set the shortest distance from 

\( p_i \) to itself equal to zero. The rest of the algorithm runs like normal Dijkstra on \( G_A \). This way, we find SSSP distances in \( G \) from \( p_i \) to all vertices in \( G_A \) in \( O(n_A \log n_A) \) time. Since \( |C_A| = O(\sqrt{n_A}) \), computing SSSP distances for all portals of \( C_A \) takes a total of \( O(n_A^{3/2} \log n_A) \) time.

The third and final problem we need to deal with is determining the order of colour intervals of \( f \cap G_{A_2} \) for each vertex of \( f \cap G_{A_1} \) where \( f \) is a face. Lemma 4 states that the order of colour intervals of a vertex of \( f \) can be computed in \( O(k \log^2 n) \) time with \( O(kn \log n) \) time for preprocessing. Looking at the proof of the lemma, we see that the preprocessing step only considers portals of \( G \), not the set of portals \( P_A \) for \( G_A \). It thus needs to be recomputed for \( G_A \). But a priori, this requires that we look at the entire graph \( G \) since shortest paths from vertices in \( G_A \) to portals in \( P_A \) need not be fully contained in \( G_A \). This will make our recursive algorithm too slow.

We modify the preprocessing step for \( G_A \) as follows. We compute \( \pi \)-pointers for each vertex of \( G_A \) and each portal of \( P_A \) as in the proof of Lemma 4 except that when finding a path to a portal \( p_i \in P_A \) we stop if we reach a portal of \( P \) (or as before, if we reach a red edge or \( p_i \)).

This modification gives \( O(k_A n_A \log n_A) \) preprocessing time where \( k_A = |P_A| \). We now show how this preprocessing can be used to efficiently determine whether \( p_i \prec_A p_j \) for two distinct portals \( p_i, p_j \in P_A \) where \( a \) is a vertex of \( G_{A_1} \) belonging to a face \( f \) of \( G \).

Let \( a' \) be the successor of \( a \) in \( f \) w.r.t. \( \prec_A \). To determine whether \( p_i \prec_A p_j \), we apply binary searches as before in paths \( \overline{P_i}(a', a) \) and \( \overline{P_j}(a', a) \). If a split is detected, or if it is detected that \( p_i \in \overline{P_j}(a', a) \) or \( p_j \in \overline{P_i}(a', a) \) then we can correctly decide if \( p_i \prec_A p_j \).

The only problem that may arise is if both binary searches end in an edge \( e \) from which there are no \( \pi \)-pointers and it still has not been determined whether \( p_i \prec_A p_j \). But in this case, one of the endpoints of \( e \) must be a portal of \( P_A \) and this portal must belong to both \( \overline{P_i}(a', a) \) and \( \overline{P_j}(a', a) \) implying that no colour intervals of \( a \) will get colour \( c_i \) or \( c_j \). Hence, it is irrelevant whether \( p_i \prec_A p_j \) and we may simply delete the two portals in the algorithm that sorts portals. When the sorting algorithm terminates, the order of the remaining portals will give the order of colour intervals of \( a \).

With the above three modifications, we get an algorithm that computes \( \delta_G(G_{A_1}, G_{A_2}) \) in \( O(k_A n_A \log^2 n_A) \) expected time. We compute \( \delta_G(G_{A_1}) \) and \( \delta_G(G_{A_2}) \) recursively where the portals for \( G_{A_1} \) are defined as those that be-
long to $P_A \cap G_A$, together with those obtained when applying the separator theorem to $G_A$ (and similarly for $G_{A_2}$).

More generally, consider a subgraph $G'$ in some node of the recursion tree. If the size of $G'$ is less than some constant, a brute-force algorithm that computes APSP-distances for vertices in $G'$ is applied to find $\delta_G(G')$ in constant time.

Otherwise, let $G_{c_1}$ and $G_{c_2}$ be the subgraphs in the two child nodes obtained by applying the separator theorem to $G'$. Let $P'$ be the set of portals of $G'$. Then for $i = 1, 2$, the set of portals of $G_{c_i}$ is defined as the union of $P' \cap G_{c_i}$ and the set of portals obtained by applying the separator theorem to $G_{c_i}$.

Let $k_p$ denote the number of portals of $G'$ that are also portals of the subgraph of $G$ belonging to the parent node of the node containing $G'$, let $k'$ be the number of additional portals of $G'$, and let $n'$ be the number of vertices of $G'$. Then similar arguments as above show that $\delta_G(G_{c_1}, G_{c_2})$ can be computed in $O((k_p + k')n' \log^2 n')$ expected time. Since $k' = O(\sqrt{n'})$, this can be rewritten as $O(k_p n' \log^2 n' + n' \sqrt{n' \log^2 n'})$.

Clearly, the sum of $O(n' \sqrt{n' \log^2 n'})$ over all non-leaf nodes of the recursion tree is $O(n^{3/2} \log^2 n)$. Let us bound the sum of $O(k_p n' \log^2 n')$ over all these nodes. Let $T(n', k_p)$ denote the total time spent in the subtree of the recursion tree rooted at a node containing a graph with $n'$ vertices and sharing $k_p$ portals with the subgraph in the parent node. Then

$$T(n', k_p) \leq T(n'/2, k_1 + c\sqrt{n'}) + T(n'/2, k_2 + c\sqrt{n'}) + O(k_p n' \log^2 n'),$$

where $c = \frac{\sqrt{2} - 1}{\sqrt{2/3}}$ and $k_1 + k_2 \leq k_p$.

It follows that the sum of all $k_p$ in the $i$th level of the recursion tree is $O(\sum_{j=0}^i 2^j \sqrt{n' / 2^j})$ so the total time spent in this level is

$$O(\sum_{j=0}^i 2^j \sqrt{n' / 2^j} n / 2^i \log^2 n') = O(n^{3/2} \log^2 n \sum_{j=0}^i 2^{j/2 - i}).$$

Since $\sum_{j=0}^i 2^{j/2 - i} \leq \sum_{j=0}^i 2^{j-i} < \sum_{j=0}^\infty 2^{-j} = 2$ and since there are $O(\log n)$ recursion levels, we have now obtained the main result of this paper.

**Theorem 6.** The maximum detour of a plane graph with $n$ vertices can be computed in $O(n^{3/2} \log^3 n)$ expected time.
7 Graphs with Bounded Treewidth

We now consider the problem of computing the maximum detour of a plane graph with bounded treewidth. The treewidth of a graph is, in a sense, a measure of the complexity of graph. The following definition and lemma are taken from [3].

**Definition 1.** A tree decomposition of a graph \( G = (V, E) \) is a pair \( (X, T) \), where \( X = \{X_i \subseteq V | i \in I\} \) is a collection of subsets of \( V \) (called bags), and a tree \( T = (I, F) \) with a node set \( I \) such that

1. \( V = \cup_{i \in I} X_i \)
2. For every edge \((u, v) \in E\) there is some bag \( X_i \in X \) such that \( u, v \in X_i \)
3. For all \( u \in V \), the nodes \( \{i \in I | u \in X_i\} \) form a connected subtree of \( T \)

The width of a tree decomposition \( (\{X_i | i \in I\}, T) \) is \( \max_{i \in I} |X_i| - 1 \). The treewidth of \( G \) is the minimum width over all tree decompositions of \( G \).

**Lemma 8.** Let \( w \geq 1 \) be a constant. Given a graph \( G = (V, E) \) with \( n > w+1 \) vertices and treewidth at most \( w \), we can find in linear time a partition of \( V \) into three subsets \( A, B, \) and \( P \) such that

1. no edge joins a vertex in \( A \) with a vertex in \( B \),
2. \( A \) and \( B \) each have between \( \frac{n}{w+1} - w \) and \( \frac{nw}{w+1} \) vertices,
3. \( P \) contains no more than \( w \) vertices, and
4. adding edges between the vertices of \( P \) does not change the treewidth.

If we apply Lemma 8 to \( G \) instead of the separator theorem by Lipton and Tarjan we get the subgraph \( G_A \) of \( G \) induced by \( A \cup P \) and the subgraph \( G_B \) of \( G \) induced by \( B \cup P \). Since the number of portals is at most \( w \), we may compute \( \delta_G(G_A, G_B) \) in \( O(wn \log^2 n) = O(n \log^2 n) \) time.

To recursively compute \( \delta_G(G_A) \), we define graph \( G' \) as the graph obtained by removing all edges of \( G \) not belonging to \( G_A \) and adding an edge between each pair of portals of \( G \). The cost of each such edge is set to the distance in \( G \) between the corresponding pair of portals. Note that the number of edges added is constant.
When we regard $G'$ as a point set, we disregard edges added between portals. Observe that for each pair of points $p$ and $q$ in $G'$, $d_{G'}(p, q) = d_G(p, q)$.

Since $G'$ has treewidth at most $w$ by Lemma 8, an inductive argument shows that $\delta_{G}(G_A) = \delta_{G'}(G_A)$ can be computed in $O(n \log^3 n)$ expected time. A similar argument shows that $\delta_{G}(G_B)$ can be computed in $O(n \log^3 n)$ expected time. This shows the second result of our paper.

**Theorem 7.** The maximum detour of a plane graph with $n$ vertices and bounded treewidth can be computed in $O(n \log^3 n)$ expected time.

### 8 Concluding Remarks

In this paper, we showed how to compute the maximum detour of a plane graph in $O(n^{3/2} \log^3 n)$ expected time. This is an improvement over the best known algorithm with $\Theta(n^2)$ running time. We also showed that if the graph has bounded treewidth, its maximum detour can be computed in $O(n \log^3 n)$ expected time.

We believe that by using parametric search as described in [1], we can obtain an algorithm computing the maximum detour of a plane graph in $O(n^{3/2} \text{polylog } n)$ worst-case time and in $O(n \text{polylog } n)$ worst-case time when the graph has bounded treewidth.

It would be interesting to try to beat the quadratic time bound also for the problem of computing the stretch factor of a planar (or plane) geometric graph. This problem appears harder since pairs of vertices achieving the maximum detour need not be co-visible.

### References


