J. Appl. Prob. Spec. Vol. **51A,** 203–212 (2014)

CELEBRATING 50 YEARS OF THE APPLIED PROBABILITY TRUST

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Part 5. Finance and econometrics

AGGREGATION OF LOG-LINEAR RISKS

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APPLIED PROBABILITY TRUST
DECEMBER 2014

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Abstract

In this paper we work in the framework of a k-dimensional vector of log-linear risks. Under weak conditions on the marginal tails and the dependence structure of a vector of positive risks, we derive the asymptotic tail behaviour of the aggregated risk and present an application concerning log-normal risks with stochastic volatility.

Keywords: Risk aggregation; log-linear model; subexponential distribution; Gumbel max-domain of attraction.

2010 Mathematics Subject Classification: Primary 60G15
Secondary 60G70

1. Introduction

In a recent contribution, Mainik and Embrechts [16] discussed important aspects of linear models of heavy-tailed risks related to risk diversification. According to that paper, a good starting point for explaining linear models is the stochastic representation of multivariate normal risks in terms of regression models. For the purpose of this introduction, we confine ourselves for the moment to two random variables (RVs). Specifically, if X_1 and X_2 are jointly normal with mean 0, variance 1 (i.e. with standard normal distribution function (DF)), and correlation $\rho \in [0, 1)$, we assume that

$$X_i = \sqrt{\rho}\, W_0 + W_{i,\rho}, \qquad W_{i,\rho} = \sqrt{1-\rho}\, W_i, \qquad i = 1, 2, \tag{1}$$

where W_0, W_1, and W_2 are independent $N(0, 1)$ RVs. Motivated by (1), we refer to the RVs $Z_1 = \mathrm{e}^{X_1}$ and $Z_2 = \mathrm{e}^{X_2}$ as *log-linear risks*. Clearly, Z_1 and Z_2 are risks with some positive dependence structure; this is a common feature of many financial and insurance risks. In many finance, insurance, and risk management applications a common model for aggregating dependent risks is the log-normal model with positive dependence; see [1, 3, 9, 12, 14, 15, 17].

The importance of this paradigm lies in the fact that on the log-scale a linear relationship such as (1) is assumed. Numerous applications based on the log-normal assumption explain the behaviour of aggregated and maximum risk. Despite the tractability and the wide applicability of log-normal-based models, the asymptotic tail behaviour of the aggregated risk $S_2 = \mathrm{e}^{X_1} + \mathrm{e}^{X_2}$ has been unknown for a long time; it was first shown in [2] that the principle of a single big jump applies for log-normal RVs defined by (1), i.e.

$$\mathbb{P}\{S_2 > u\} \sim \mathbb{P}\{\max(\mathrm{e}^{X_1}, \mathrm{e}^{X_2}) > u\} \sim 2\mathbb{P}\{X_1 > \ln u\} \quad \text{as } u \to \infty.$$

We use the standard notation $a(u) \sim b(u)$ to mean $\lim_{u\to\infty} a(u)/b(u) = 1$ for nonnegative functions $a(\cdot)$ and $b(\cdot)$.

In this paper, instead of making specific distributional assumptions on the W_i we shall impose only weak conditions on the marginal tails and on the joint dependence structure. For instance, for the simple setup of bivariate log-linear risks, the main finding of this contribution is that, under such assumptions, the asymptotic tail behaviour of the aggregated risk S_2 is still determined by the tail asymptotics of Z_1 and Z_2. In the special case when $W_i,\ i = 1, 2$, satisfy

$$\lim_{x\to\infty} \frac{\mathbb{P}\{W_i > x\}}{\mathbb{P}\{W > x\}} = c_i \in (0, \infty), \tag{2}$$

where W is an $N(0, 1)$ RV, the tail asymptotics of S_2 are determined by those of W.

The paper is organised as follows. Section 2 consists of some preliminary results. In Section 3 we present our main result. In Section 4 we discuss a log-normal model with stochastic volatility. Proofs of our results are relegated to Section 5.

2. Preliminaries

In this section we briefly discuss some classes of univariate DFs characterised by their tail behaviour. Additionally, we present two results on the tail asymptotics of products and sums of independent RVs.

A well-known class of univariate DFs with numerous applications to risk aggregation is that of *subexponential DFs*; see [6, 11]. An RV U and its DF F with support on $(0, \infty)$ are called subexponential if

$$\lim_{u\to\infty} \mathbb{P}\{U + U^* > u\} \sim 2\mathbb{P}\{U > u\} \quad \text{as } u \to \infty,$$

where U^* is an independent copy of U. It is well known (see, e.g. [10]) that a subexponential DF F is *long tailed*, i.e. there exists some positive measurable function $a(\cdot)$ such that $\lim_{u\to\infty} a(u) = \lim_{u\to\infty} u/a(u) = \infty$ and, furthermore,

$$\mathbb{P}\{U > u + a(u)\} \sim \mathbb{P}\{U > u\} \quad \text{as } u \to \infty.$$

Canonical examples of such DFs are the Pareto and the log-normal. A further interesting example is the Weibull DF with tail DF $\overline{F}(x) = 1 - F(x) = \exp(-x^{\beta}),\ \beta \in (0, 1)$, for which we can choose

$$a(u) = o(u^{1-\beta}) \quad \text{as } u \to \infty.$$

An important property of log-normal and Weibull RVs is that their DFs are in the Gumbel max-domain of attraction (MDA) with some positive scaling function $b(\cdot)$, meaning that

$$\lim_{u\to\infty} \frac{\mathbb{P}\{U > u + tb(u)\}}{\mathbb{P}\{U > u\}} = \exp(-t), \qquad t > 0. \tag{3}$$

Condition (3) is equivalent to the fact that the RV U has its DF in the Gumbel MDA; see [6, 20]. We shall abbreviate the limit relation (3) to $U \in \mathrm{GMDA}(b)$. RVs satisfying (3) have rapidly varying tail DF $\overline{F}$, i.e. for any $\lambda > 1$,

$$\lim_{u\to\infty} \frac{\overline{F}(\lambda u)}{\overline{F}(u)} = 0. \tag{4}$$

The next lemma is crucial for the derivation of the asymptotic tail behaviour of the product of two independent nonnegative RVs.

Lemma 1. *Let X, Y, and Y^* be nonnegative RVs with infinite right endpoint and such that X is independent of (Y, Y^*). If the tail DF $\overline{F}$ of X satisfies (4) for some $\lambda_0 > 1$ and further $\lim_{u\to\infty} \mathbb{P}\{Y > u\}/\mathbb{P}\{Y^* > u\} = c \in (0, \infty)$, then*

$$\mathbb{P}\{XY > u\} \sim c\,\mathbb{P}\{XY^* > u\} \quad \text{as } u \to \infty. \tag{5}$$

Remark 1. (a) The class of DFs F with infinite right endpoint and such that (4) holds for some $\lambda_0 > 1$ strictly contains the class of DFs with a rapidly varying tail. For $\overline{F}$ rapidly varying, (5) was proved in [22, Lemma A.5].

(b) Note that condition (4) on $X = \mathrm{e}^W$ holds if and only if the RV W has an infinite right endpoint and, for some $\eta > 0$,

$$\lim_{u\to\infty} \frac{\mathbb{P}\{W > u + \eta\}}{\mathbb{P}\{W > u\}} = 0. \tag{6}$$

Lemma 1 implies the following result for convolutions.

Corollary 1. *Let $W_1, \ldots, W_k$ be independent RVs with infinite right endpoints. Suppose that condition (6) holds for each W_i with a suitable constant $\eta_i > 0$, $i = 1, \ldots, k$. Furthermore, if the independent RVs $V_1, \ldots, V_k$ satisfy $\lim_{u\to\infty} \mathbb{P}\{W_i > u\}/\mathbb{P}\{V_i > u\} = p_i \in (0, \infty)$, $i \le k$, then, for any positive constants $\theta_1, \ldots, \theta_k$,*

$$\mathbb{P}\bigg\{\sum_{i=1}^{k} \theta_i W_i > u\bigg\} \sim \bigg(\prod_{i=1}^{k} p_i\bigg) \mathbb{P}\bigg\{\sum_{i=1}^{k} \theta_i V_i > u\bigg\} \quad \textit{as } u \to \infty. \tag{7}$$

Remark 2. If $W_1, \ldots, W_k$ are independent RVs satisfying $\mathbb{P}\{W_i > u\} \sim p_i u^{\alpha_i} \mathrm{e}^{-u^2/2}$ as $u \to \infty$ for some $\alpha_i \in \mathbb{R}$ and $p_i \in (0, \infty)$, $i \le k$, then $W_i \in \mathrm{GMDA}(b)$ with $b(u) = 1/u$; see, e.g. [6, p. 155]. Consequently, (7) holds for any positive $\theta_1, \ldots, \theta_k$ and independent RVs $V_1, \ldots, V_k$ such that each V_i has a density f_i satisfying $f_i(u) \sim u^{\alpha_i+1} \mathrm{e}^{-u^2/2}$ as $u \to \infty$. By Theorem 1.1 of [21] we obtain

$$\mathbb{P}\bigg\{\sum_{i=1}^{k} \theta_i V_i > u\bigg\} \sim (\sqrt{2\pi})^{k-1} \sigma^{1-2k-2\alpha} \prod_{i=1}^{k} \theta_i^{\alpha_i+1} u^{\alpha+k-1} \mathrm{e}^{-u^2/(2\sigma^2)} \quad \text{as } u \to \infty, \tag{8}$$

where $\alpha = \sum_{i=1}^{k} \alpha_i$ and $\sigma^2 = \sum_{i=1}^{k} \theta_i^2$. The asymptotic expansion (8) is shown in [18, Lemma 8.6]; see also [8, Theorem 2.2].

3. Main result

Motivated by (1), we introduce a k-dimensional *log-linear model* of positive risks. For this purpose, let W_i, $i = 0, \ldots, k$, be independent RVs. For nonnegative $\rho \in [0, 1)$, write $W_{i,\rho} = \sqrt{1-\rho}\, W_i$. Now introduce the linearly dependent RVs $X_i = \sqrt{\rho_0}\, W_0 + W_{i,\rho_i}$ for constants $\rho_0 > 0$ and $\rho_i \in [0, 1)$. Define the log-linear model for positive constants θ_i by

$$Z_i = \theta_i \mathrm{e}^{X_i}, \qquad i = 1, \ldots, k. \tag{9}$$

In the credit risk literature, model (9) with independent and identically distributed (i.i.d.) standard normal $W_0, \ldots, W_k$ is usually referred to as the one-factor (or Vasicek) model and forms the mathematical basis underlying the CreditMetrics™/KMV approach; see, for instance, [5, Section 2.5].

In Theorem 1 below, an explicit expansion for the tail of the aggregated risk is derived by assuming subexponentiality of certain factors, which in particular implies that the aggregated risk S_k and maximum risk M_k are tail equivalent, where

$$S_k := \sum_{i=1}^{k} Z_i, \qquad M_k := \max_{1\le i\le k} Z_i.$$

As in the log-normal case investigated by Asmussen and Rojas-Nandayapa [2], the principle of a single big jump also applies in our framework; see [11] for an insightful explanation of this phenomenon. In fact, the asymptotic tail behaviour of the aggregated risk is controlled by the base risk W_0 and the index set

$$J = \{1 \le j \le k : \rho_j = \varrho\}, \quad \text{where} \quad \varrho := \min_{1\le i\le k} \rho_i.$$

Theorem 1. *Consider the log-linear model $Z_1, \dots, Z_k$ defined by (9). Assume in addition that the following conditions hold.*

(i) *The RVs W_i satisfy the tail equivalence condition (2) for an RV W and positive constants $c_i,\ i = 1, \dots, k$.*

(ii) *The RV W satisfies (6) for some $\eta > 0$, and $\mathrm{e}^{\sqrt{1-\varrho}\,W}$ is subexponential.*

(iii) *The RV W_0 satisfies (6) for some $\eta = \eta_0 > 0$.*

Then, as $u \to \infty$,

$$\mathbb{P}\{S_k > u\} \sim \mathbb{P}\{M_k > u\} \sim \sum_{i\in J} c_i\, \mathbb{P}\biggl\{\sqrt{\rho_0}\, W_0 + \sqrt{1-\varrho}\, W^* > \ln\biggl(\frac{u}{\theta_i}\biggr)\biggr\},$$

where W_0 and W^* are independent, and W^* is an independent copy of W.

Furthermore, if (6) holds for W and W_0 and some η satisfying

$$0 < \eta \le (1-\varrho)^{-1/2} \min_{\{i\in J:\ \theta_i<\widetilde{\theta}\}} \ln\biggl(\frac{\widetilde{\theta}}{\theta_i}\biggr), \tag{10}$$

where $\widetilde{\theta} = \max_{i\in J} \theta_i$, and the right-hand side of (10) equals ∞ if $\theta_i = \widetilde{\theta}$ for all i, then

$$\mathbb{P}\{S_k > u\} \sim \mathbb{P}\{M_k > u\} \sim \sum_{\{i\in J:\ \theta_i=\widetilde{\theta}\}} c_i\, \mathbb{P}\biggl\{\sqrt{\rho_0}\, W_0 + \sqrt{1-\varrho}\, W^* > \ln\biggl(\frac{u}{\widetilde{\theta}}\biggr)\biggr\}.$$

Theorem 1 can also be formulated to cover differences of log-linear risks by allowing some θ_i to be negative. Under the assumptions of Theorem 1, if $\widetilde{\theta} > 0$ then any i such that $\theta_i < 0$ does not belong to the index set J.

In general, Theorem 1 does not follow from the results in [9] because we do not impose conditions on the hazard rate function. In this context, we also mention the recent contribution [12], in which the asymptotic tail behaviour of the differences of log-normal risks is investigated.

Example 1. If $W \in \mathrm{GMDA}(b)$ with scaling function $b(u) = 1/u$ then $\mathrm{e}^{\sqrt{1-\varrho}\,W} \in \mathrm{GMDA}(b^*)$ with $b^*(u) = (1-\varrho)u/\ln u$ and $\mathrm{e}^{\sqrt{1-\varrho}\,W}$ is subexponential by virtue of the Goldie–Resnick condition; see, e.g. [6, p. 149]. In particular, if V is an $N(0,1)$ RV with density φ then $V \in \mathrm{GMDA}(b)$ with $b(u) = 1/u$. Hence, for W_0 and W with tail behaviour proportional to that of the standard normal RV V, i.e. for positive ν and ν_0,

$$\mathbb{P}\{W_0 > u\} \sim \nu_0\, \mathbb{P}\{V > u\}, \qquad \mathbb{P}\{W > u\} \sim \nu\, \mathbb{P}\{V > u\}, \quad \text{as } u \to \infty, \tag{11}$$

the conditions of Theorem 1 are satisfied. In view of Corollary 1, for an independent copy V^* of V and with $\sigma = \sqrt{1+\rho_0-\varrho}$,

$$\begin{aligned} \mathbb{P}\{\sqrt{\rho_0}\, W_0 + \sqrt{1-\varrho}\, W^* > u\} &\sim \nu_0\nu\, \mathbb{P}\{\sqrt{\rho_0}\, V + \sqrt{1-\varrho}\, V^* > u\} \\ &= \nu_0\nu\, \mathbb{P}\{\sigma V > u\} \\ &\sim \nu_0\nu\, \frac{\sigma}{u}\varphi\biggl(\frac{u}{\sigma}\biggr) \quad \text{as } u \to \infty. \end{aligned}$$

Thus, we have derived the following result under the assumptions of Theorem 1 and the additional conditions (10) and (11):

$$\mathbb{P}\{S_k > u\} \sim \mathbb{P}\{M_k > u\} \sim \nu_0\, \nu \sum_{j\in J} c_j \,\frac{\sigma}{\ln(u/\widetilde{\theta}\,)}\, \varphi\Big(\frac{\ln(u/\widetilde{\theta}\,)}{\sigma}\Big) \quad \text{as } u \to \infty.$$

We mention in passing that, under the assumptions above, each RV $e^{W_{i,\rho_i}}$ has DF in the Gumbel MDA; hence, the approach suggested in [17] is applicable.

4. Log-normal risks with stochastic volatility

We now discuss the log-normal model with stochastic volatility. Consider the log-linear model of the previous section, assuming now that the W_i are conditionally independent normal RVs with zero means and stochastic volatility $I_i > 0$, $i = 0, \dots, k$. This means we have the representation

$$W_i = I_i\, Y_i, \qquad i = 0, \dots, k,$$

where Y_i and I_i, $i = 0, \dots, k$, are independent RVs, and (Y_i) is an i.i.d. $N(0, 1)$ sequence. We note that there is a close relationship between our model and normal variance mixture models; see, e.g. [4].

In a practical setting, the I_i can be understood as random deflators. Therefore, we assume that the I_i are supported on $(0, 1]$ with an atom at 1, i.e. for every $i \le k$, there exists $c_i = \mathbb{P}\{I_i = 1\} > 0$. The asymptotic tail behaviour of the W_i in this model is very close to that of the Y_i; see Lemma 2 in Appendix A.

4.1. Maximum and aggregated risk

The log-normal model with stochastic volatility, defined for given constants $\theta_1 \ge \cdots \ge \theta_k > 0$ by

$$Z_i = \theta_i \exp(\sqrt{\rho_0}\, I_0 Y_0 + \sqrt{1-\rho_i}\, I_i Y_i), \qquad i = 1, \dots, k,$$

is of special interest because it facilitates the incorporation of random deflation effects. The positive weights θ_i correspond to a deterministic trend $\ln \theta_i$ in the log-linear relationship for Z_i.

In view of Lemma 2 in Appendix A,

$$\mathbb{P}\{W_i > u\} \sim c_i \frac{\varphi(u)}{u} \quad \text{as } u \to \infty,\ i = 0, \dots, k,$$

where φ is the density of an $N(0, 1)$ RV. Applying Example 1 and assuming for simplicity that $\varrho = \rho_1$ and $\theta_i = 1$ for all $i \in J$, we obtain, for fixed $k \ge 1$,

$$\mathbb{P}\{S_k > u\} \;\sim\; \mathbb{P}\{M_k > u\} \;\sim\; c_0 \sum_{j\in J} c_j \,\frac{\sigma}{\ln u}\varphi\Big(\frac{\ln u}{\sigma}\Big) \quad \text{as } u \to \infty, \tag{12}$$

where $\sigma = \sqrt{1+\rho_0-\rho_1}$.

4.2. Asymptotic behaviour of VaR and CTE

Since the Y_i have continuous DFs, the Z_i have continuous DFs as well. Hence, by definition, the conditional tail expectation (also referred to as expected shortfall) of S_k is given by

$$\mathrm{CTE}_q(S_k) = \mathbb{E}\{S_k \mid S_k > \mathrm{VaR}_q(S_k)\},$$

where $q \in (0, 1)$ is a predefined confidence level and $\mathrm{VaR}_q(X)$ is the value at risk at level q for the risk X, i.e. $\mathrm{VaR}_q(X) = \inf\{s \in \mathbb{R} \colon \mathbb{P}\{X \le s\} \ge q\}$.

Relation (12) implies that

$$\mathbb{E}\{(S_k - u) \mid S_k > u\} \sim b(u) \sim \frac{\sigma^2 u}{\ln u} \quad \text{as } u \to \infty;$$

see, e.g. [6]. Since $v_n = \mathrm{VaR}_{1-1/n}(S_k) \to \infty$ as $n \to \infty$, we obtain

$$\begin{aligned} \mathrm{CTE}_{1-1/n}(S_k) &= \mathbb{E}\{(S_k - v_n) \mid S_k > v_n\} + v_n \\ &= v_n\left(\frac{\sigma^2}{\ln v_n}(1 + o(1)) + 1\right) \\ &\sim \mathrm{VaR}_{1-1/n}(S_k) \quad \text{as } n \to \infty. \end{aligned} \tag{13}$$

In practice, the level $q = 1 - 1/n$ is fixed with n typically large, leading to confidence levels 0.95, 0.99, 0.995, 0.999, and even 0.9997. For instance, the capital charge for credit and operational risk is calculated with a confidence level of $q = 0.999$ and a holding period (horizon) of one year. For the calculation of economic capital, one typically takes $q = 0.9997$. Although CTE_q is more conservative than VaR_q, (13) implies that their asymptotic behaviour (for q close to 1) is similar. The conclusion is that, in terms of the asymptotic behaviour of VaR and CTE, this model is similar to the log-normal model.

Remark 3. These properties of VaR and CTE link up very much with the recent discussion around the Basel Committee's regulatory 2012 document *Fundamental Review of the Trading Book* (see http://www.bis.org/publ/bcbs219.htm). On page 41 of this document, Question 8 reads:

> What are the likely constraints with moving from Value-at-Risk (VaR) to Expected Shortfall (ES = CTE), including any challenges in delivering robust backtesting and how might these be best overcome?

The phrase 'moving from' has to be interpreted as 'using ES as an alternative risk measure to VaR for setting capital adequacy standards'. An important aspect of this discussion concerns the understanding of portfolio structures where it does not matter much (hence, Section 4.2), and, more importantly, those for which significant differences do exist. For a discussion on the latter, see, for instance, [19]. For results on risk measure estimation and model uncertainty (mainly at the level of VaR), see [7].

5. Proofs

Proof of Lemma 1. Let F and G denote the DFs of X and Y, respectively. Since X and Y are independent, for given constants δ and γ for which $0 < \delta < \gamma < \infty$ and any $u > 0$, $\mathbb{P}\{XY > u\}$ equals

$$\int_0^\delta \mathbb{P}\left\{X > \frac{u}{y}\right\} \mathrm{d}G(y) + \int_\delta^\gamma \mathbb{P}\left\{X > \frac{u}{y}\right\} \mathrm{d}G(y) + \int_\gamma^\infty \mathbb{P}\left\{X > \frac{u}{y}\right\} \mathrm{d}G(y).$$

Choosing γ such that $\gamma/\delta > \lambda_0$, condition (4) for $\lambda = \lambda_0$ implies that, as $u \to \infty$,

$$\frac{\int_0^\delta \mathbb{P}\{X > u/y\}\,\mathrm{d}G(y)}{\int_\gamma^\infty \mathbb{P}\{X > u/y\}\,\mathrm{d}G(y)} \le \frac{\mathbb{P}\{X > u/\delta\}}{\mathbb{P}\{X > u/\gamma\}\mathbb{P}\{Y > \gamma\}} \to 0.$$

Since, for any $u > 0$, we also have $\mathbb{P}\{XY > u\} \geq \int_\delta^\infty \mathbb{P}\{X > u/y\}\,\mathrm{d}G(y)$ (cf. the denominator on the left-hand side above), we conclude that, for any fixed $\delta > 0$,

$$\mathbb{P}\{XY > u\} \sim \int_\delta^\infty \mathbb{P}\left\{X > \frac{u}{y}\right\} \mathrm{d}G(y) \quad \text{as } u \to \infty.$$

Next, integrating by parts,

$$\begin{aligned}\int_\delta^\infty \mathbb{P}\left\{X > \frac{u}{y}\right\} \mathrm{d}G(y) &= \overline{F}\left(\frac{u}{\delta}\right)\mathbb{P}\{Y > \delta\} + \int_0^{u/\delta} \mathbb{P}\left\{Y > \frac{u}{y}\right\} \mathrm{d}F(y) \\ &= (1+o(1)) \int_0^{u/\delta} \mathbb{P}\left\{Y^* > \frac{u}{y}\right\}\left(\frac{\mathbb{P}\{Y > u/y\}}{\mathbb{P}\{Y^* > u/y\}} - c\right)\mathrm{d}F(y) \\ &\quad + (1+o(1))c \int_0^{u/\delta} \mathbb{P}\left\{Y^* > \frac{u}{y}\right\} \mathrm{d}F(y).\end{aligned}$$

The first term on the right-hand side is of smaller order than the second because $\lim_{u\to\infty} \mathbb{P}\{Y > u\}/\mathbb{P}\{Y^* > u\} = c$ and we can choose $\delta > 0$ arbitrarily large. Furthermore, integrating by parts again and arguing as in the first part of the proof shows that the second term has the asymptotic order

$$c\int_0^{u/\delta} \mathbb{P}\left\{Y^* > \frac{u}{y}\right\} \mathrm{d}F(y) \sim c\,\mathbb{P}\{XY^* > u\}$$

for any fixed $\delta > 0$. This completes the proof.

Proof of Corollary 1. In what follows, assume without loss of generality that all $\theta_i = 1$ (we can make this assumption because W_i satisfies (6) for $\eta = \eta_i$ if and only if $\theta_i W_i$ does so for $\eta = \eta_i/\theta_i$). We prove the result by induction on k. For $k = 1$, the result is just a consequence of the condition $\lim_{u\to\infty} \mathbb{P}\{W_1 > u\}/\mathbb{P}\{V_1 > u\} = p_1$. Let (V_i') be a copy of (V_i) which is independent of (W_i).

For $k = 2$, take $X = \mathrm{e}^{W_2}$, $Y = \mathrm{e}^{W_1}$, and $Y^* = \mathrm{e}^{V_1'}$. In view of (6) for W_2 the assumptions of Lemma 1 are satisfied. Therefore,

$$\mathbb{P}\{W_1 + W_2 > u\} = \mathbb{P}\{XY > \mathrm{e}^u\} \sim p_1\mathbb{P}\{XY^* > \mathrm{e}^u\} = \mathbb{P}\{V_1' + W_2 > u\}.$$

Next choose $X = \mathrm{e}^{V_1'}$, $Y = \mathrm{e}^{W_2}$, and $Y^* = \mathrm{e}^{V_2'}$, and apply Lemma 1 to obtain (7) for $k = 2$. Note that we also used the fact that (6) holds for V_1'.

Assume that (7) holds for $k = n \geq 2$. In view of the proof above we may also assume that we proved that, for $u \to \infty$,

$$\prod_{i=2}^n p_i\,\mathbb{P}\{(V_2' + \cdots + V_n') + W_{n+1} > u\} \sim \mathbb{P}\{(W_2 + \cdots + W_n) + W_{n+1} > u\}. \tag{14}$$

Take $X = \mathrm{e}^{W_{n+1}}$, $Y = \mathrm{e}^{W_1+\cdots+W_n}$, and $Y^* = \mathrm{e}^{V_1'+\cdots+V_n'}$. By the induction hypothesis and (6) for W_{n+1}, the assumptions of Lemma 1 are satisfied. Therefore,

$$\begin{aligned}\mathbb{P}\{(W_1 + \cdots + W_n) + W_{n+1} > u\} &= \mathbb{P}\{XY > \mathrm{e}^u\} \\ &\sim \prod_{i=1}^n p_i\mathbb{P}\{XY^* > \mathrm{e}^u\} \\ &= \prod_{i=1}^n p_i\mathbb{P}\{(V_1' + \cdots + V_n') + W_{n+1} > u\}.\end{aligned} \tag{15}$$

Now choose $X = e^{V_1'}$, $Y = e^{(V_2'+\cdots+V_n')+W_{n+1}}$, and $Y^* = e^{(W_2+\cdots+W_n)+W_{n+1}}$, and again apply Lemma 1 to obtain

$$\prod_{i=1}^n p_i\,\mathbb{P}\{V_1' + (V_2' + \cdots + V_n') + W_{n+1} > u\} \sim p_1\,\mathbb{P}\{V_1' + (W_2 + \cdots + W_n) + W_{n+1} > u\}$$
$$\sim \mathbb{P}\bigg\{\sum_{i=1}^{n+1} W_i > u\bigg\},$$

using also (14). Together with (15) this proves (7) for $k = n + 1$.

Proof of Theorem 1. First assume that the index $i \le k$ is such that $\rho_i > \varrho = \min_{1\le j\le k}\rho_j$. Then, for sufficiently large u,

$$\begin{aligned}
\mathbb{P}\{\theta_i e^{W_{i,\rho_i}} > u\} &\le \mathbb{P}\{\widetilde{\theta} e^{W_{i,\rho_i}} > u\}\\
&= \mathbb{P}\bigg\{W_i > \ln\bigg(\frac{u}{\widetilde{\widetilde{\theta}}}\bigg)(1-\varrho_i)^{-0.5}\bigg\}\\
&\sim c_i\,\mathbb{P}\bigg\{W > \ln\bigg(\frac{u}{\widetilde{\widetilde{\theta}}}\bigg)(1-\varrho)^{-0.5} + ((1-\rho_i)^{-0.5} - (1-\varrho)^{0.5})\ln\bigg(\frac{u}{\widetilde{\widetilde{\theta}}}\bigg)\bigg\}\\
&\le c_i\,\mathbb{P}\bigg\{W > \ln\bigg(\frac{u}{\widetilde{\widetilde{\theta}}}\bigg)(1-\varrho)^{-0.5} + \eta\bigg\}\\
&= o(\mathbb{P}\{\widetilde{\theta} e^{\sqrt{1-\varrho}\,W} > u\}) \quad \text{as } u \to \infty,
\end{aligned} \tag{16}$$

where in the last step we have used (6) for any choice of $\eta > 0$.

Next consider an index $i \in J$. An argument similar to that above shows that

$$\begin{aligned}
\mathbb{P}\{\theta_i e^{W_{i,\rho_i}} > u\} &= \mathbb{P}\bigg\{W_i > \ln\bigg(\frac{u}{\theta_i}\bigg)(1-\varrho)^{-0.5}\bigg\}\\
&\sim c_i\,\mathbb{P}\bigg\{W > \ln\bigg(\frac{u}{\theta_i}\bigg)(1-\varrho)^{-0.5}\bigg\} \quad \text{as } u \to \infty,
\end{aligned}$$

and, under the additional assumption (10), a similar argument also proves (16) for $\theta_i < \widetilde{\theta}$. The subexponentiality of $e^{\sqrt{1-\varrho}\,W}$ and Corollary 3.19 of [11] imply that

$$\begin{aligned}
\mathbb{P}\bigg\{\sum_{i=1}^k \theta_i e^{W_{i,\rho_i}} > u\bigg\} &\sim \sum_{i=1}^k \mathbb{P}\{\theta_i e^{W_{i,\rho_i}} > u\}\\
&\sim \sum_{i\in J} c_i\,\mathbb{P}\{\theta_i e^{\sqrt{1-\varrho}\,W} > u\} \quad \text{as } u \to \infty.
\end{aligned} \tag{17}$$

Furthermore, under the additional condition (10), the right-hand side is equivalent to

$$\sum_{\{i\in J\,:\,\theta_i=\widetilde{\theta}\}} c_i\,\mathbb{P}\{\widetilde{\theta}\, e^{\sqrt{1-\varrho}\,W} > u\}.$$

By assumption, W_0 satisfies (6) for some $\eta = \eta_0 > 0$; hence, $X = e^{\sqrt{\rho_0}\,W_0}$ satisfies (4) for some $\lambda_0 > 1$. Write $Y = \sum_{i=1}^k \theta_i e^{W_{i,\rho_i}}$ and interpret (17) as the right tail of an RV $Y^* = \Theta e^{\sqrt{1-\varrho}\,W^*}$, where W^* is a copy of W independent of W_0 and Θ is an RV independent of W^*; then, from Lemma 1, it follows that

$$\mathbb{P}\{XY > u\} \sim \mathbb{P}\{XY^* > u\} = \sum_{i\in J} c_i\,\mathbb{P}\{\theta_i e^{\sqrt{\rho_0}\,W_0 + \sqrt{1-\varrho}\,W^*} > u\} \quad \text{as } u \to \infty.$$

In view of Bonferroni's inequalities, we also have

$$\mathbb{P}\Big\{\max_{1\le i\le k}\theta_i \mathrm{e}^{W_{i,\rho_i}}>u\Big\}\sim\sum_{i=1}^{k}\mathbb{P}\{\theta_i \mathrm{e}^{W_{i,\rho_i}}>u\}\sim\sum_{i\in J}c_i\ \mathbb{P}\{\theta_i \mathrm{e}^{\sqrt{1-\varrho}\,W}>u\}\quad\text{as } u\to\infty.$$

Again applying Lemma 1 for $Y=\max_{1\le i\le k}\theta_i \mathrm{e}^{W_{i,\rho_i}}$ and $Y^*=\Theta \mathrm{e}^{\sqrt{1-\varrho}\,W^*}$, we obtain

$$\mathbb{P}\Big\{\mathrm{e}^{\sqrt{\rho_0}\,W_0}\max_{1\le i\le k}\theta_i \mathrm{e}^{W_{i,\rho_i}}>u\Big\}\sim\sum_{i\in J}c_i\ \mathbb{P}\{\theta_i \mathrm{e}^{\sqrt{\rho_0}\,W_0+\sqrt{1-\varrho}\,W^*}>u\}\quad\text{as } u\to\infty.$$

This completes the proof.

Appendix A

We give here the tail asymptotics of deflated risks, assuming that the deflator is bounded and the risk has a rapidly varying tail. For the case that the deflator is bounded and has a regularly varying tail at the right endpoint of the DF, see, e.g. [13].

Lemma 2. *Consider independent RVs I and W such that I is supported on* $(0,1]$ *and the tail of W is rapidly varying. Then*

$$\lim_{u\to\infty}\frac{\mathbb{P}\{IW>u\}}{\mathbb{P}\{W>u\}}=\mathbb{P}\{I=1\}.$$

Proof. For any $u>0$ and $z\in(0,1)$,

$$\mathbb{P}\{IW>u\}=\mathbb{P}\{IW>u,\ I\in(0,z]\}+\mathbb{P}\{IW>u,\ I\in(z,1)\}+\mathbb{P}\{IW>u,\ I=1\}.$$

Observe that $\mathbb{P}\{IW>u, I=1\}=\mathbb{P}\{W>u\}\mathbb{P}\{I=1\}$,

$$\lim_{z\uparrow1}\limsup_{u\to\infty}\frac{\mathbb{P}\{IW>u,\ I\in(z,1)\}}{\mathbb{P}\{W>u\}}\le\lim_{z\uparrow1}\mathbb{P}\{I\in(z,1)\}=0,$$

and, by rapid variation,

$$\limsup_{u\to\infty}\frac{\mathbb{P}\{IW>u,\ I\in(0,z]\}}{\mathbb{P}\{W>u\}}\le\lim_{u\to\infty}\frac{\mathbb{P}\{W>uz^{-1}\}}{\mathbb{P}\{W>u\}}=0.$$

This completes the proof.

Acknowledgements

EH was partially supported by SNSF 200021-140633/1 and FP7 M.C. IRSESF RARE - 318984. For TM, part of this research was done at RiskLab, ETH Zürich; he acknowledges the Forschungsinstitut für Mathematik (FIM) for financial support.

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