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Abstract

The Taylor approximation to the general n*-input constant elasticity of substitution (CES) function is presented and compared to Kmenta's well-known result for* n=*2. It is shown that the approximation to the* n*-input CES function is a translog function as for* n=*2, but that the restrictions on the translog parameters are more complex than in the general case. Bias and consistency of the Taylor approximation to the general* n*-input CES function are discussed and it is argued that the approximation will only give reliable results for a very limited regime of CES parameters and input values.*

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1. INTRODUCTION

Within fishery economics much attention is currently being paid to establishing production relationships between landings (weight or value) and effort exerted, such as days at sea, number of crew members, vessel characteristics etc. Proper understanding of such relationships is important when discussing relevant management initiatives with the aim to sustain the fish resource.

Investigation of production relationships divides into two tracks, non-parametric and parametric. The former focuses on the Data Envelopment Analysis (DEA), which is gaining increasing attention as a non-parametric tool for estimating excess capacity in fisheries. The latter, the parametric approach, focuses on production functions and production frontiers, the former of which is addressed in this paper.

A production function assumes a parametric functional relationship between output (landings) y and input effort vector x and, if available, fish stock B, $y=f(x,B)$. Early investigations of such relationships dates back to Shaefer (1957) who proposed a linear relationship between catch (y) as the dependent variable and effort (E) and fish stock (B) as the explanatory variables, i.e. a relationship $y=qEB$, where q is the catchability coefficient. This form has in later work been recognised as being rather restrictive (Hanneson, 1983), and the Schaefer form has thus been extended to several different forms, of which the Cobb-Douglas (CD), the Constant Elasticity of Substitution (CES) and the translog are all well-known and widely employed. Of these the CD form is a special case of the CES form. The interest for these two forms is associated with their simplicity, the straightforward interpretation of the parameters of the functions and hence with their direct applicability in policy matters (Varian, 1992).

The translog function is more general than the CD and CES, as it allows for varying returns to scale and varying factor elasticity of substitution. The translog form may generally be viewed as a second order Taylor approximation to an arbitrary production form (Heathfield and Wibe, 1987), and does as such cover a wide variety of production functions, the reason why the translog function is gaining increased attention and is widely employed.

While the translog form and the logarithm of the CD are both linear, the CES form is on the contrary non-linear and cannot be linearised analytically. Estimating functional parameters for the CES function thus includes non-linear fitting techniques, which are generally complicated.

Kmenta (1967A) presents a linear approximation to the two-input CES function, employing Taylor approximation. His result is a translog form fulfilling certain restrictions on the translog parameters. Kmenta's approximation is widely applicable as the two-input case is often encountered in production theory, the inputs being capital and labour.

Within fishery economics more than two inputs are however often present, as the effort may include several different factors. It is thus often necessary to consider the *n*-input forms of the above-described production functions[2], such as the CD, the CES and the translog form. In this respect the question has been raised whether the Kmenta translog approximation to the two-input CES function is directly applicable in the *n*-input case. As it has not been possible to find any literature on this subject, a general analysis of the Taylor approximation to the *n*-input CES function has been performed, and it has been shown that the result is a translog function, as in the two-input case, but with more general restrictions on the translog parameters than for the two input case. This general result is of course consistent with Kmenta's result when *n* is set equal to *2*.

The purpose of this paper is to present this Taylor approximation to the general *n*-input CES function. A short introduction to two- respectively *n*-input elasticities of substitution and to the two respectively *n*-input CES functions will be given, followed by at short presentation of Kmenta's result, leading to the core result of the paper, the first order Taylor approximation to the *n*-input CES function, the proof of which may be found in the appendix.

The paper is concluded with at discussion of the bias and consistency of the general translog approximation to the CES function, and it will be shown that the approximation only gives reasonable results in a very limited regime of the CES parameters as

[2] This has e.g. been the case in the EU-project 'The Relationship Between Fleet Capacity, Landings, and the Component Parts of Fishing Effort' (DGI-FISH, Study 99/65), for which the general purpose has been to fit different production forms to observed landings and employed effort, where the effort include several different factors, such as days at sea, maximum horsepower, number of crew members etc.

well as range of input values, thus indicating that caution should generally be exerted when employing approximations to the CES function.

The paper is divided into 7 sections and one appendix. Section 2 gives a general introduction to the concept of elasticity of substitution, and discusses the two-input case against the multiple input case. Section 3 presents the constant elasticity of substitution (CES) functions for the two-input respectively the multiple input case. Section 4 presents Kmenta's translog approximation to the two-input CES function, while section 5 gives the main result of the paper, namely the general translog approximation to the multiple input CES function. Moreover this section presents an alternative and easier, but still linear, testing procedure. Section 6 presents the results of an analysis of bias and consistency of the translog approximation to the CES function, while section 7 sums up the work.

2. ELASTICITY OF SUBSTITUTION

The degree of substitutability between the input factors of a production is an essential concept within production theory. Hicks (1932) was the first to introduce and discuss a dimensionless measure of substitutability of the input factors, the so-called elasticity of substitution, for a two-factor production. The Hicks elasticity of substitution is defined as the relative change in the proportion of the two input factors as a function of the relative change of the corresponding marginal rate of technical substitution. The elasticity of substitution for a two-input technology is as such defined as:

$$(1) \quad \sigma = \left(\frac{d(x_2/x_1)}{x_2/x_1} \right) \Bigg/ \left(\frac{d(MP_1/MP_2)}{MP_1/MP_2} \right) = \frac{d \ln(x_2/x_1)}{d \ln(MP_1/MP_2)}$$

where x_1 and x_2 are the input values and MP_1 and MP_2 are the marginal products of the technology. σ is thus the rate at which the relative input proportion will change given a small relative change in the rate of which the two inputs must substitute for each other in order to keep output fixed.

It is straightforward to show that when the technology is described by the production function $y=f(x_1,x_2)$, equation (1) reduces to:

$$(2) \quad \sigma = \frac{(x_1 f_1 + x_2 f_2)}{x_1 x_2} \frac{f_1 f_2}{(2 f_{12} f_1 f_2 - f_{11} f_2^2 - f_{22} f_1^2)} = \frac{(x_1 f_1 + x_2 f_2)}{x_1 x_2} \frac{f_1 f_2}{|H|}$$

where f_i is the i'th derivative of f ($\partial f / \partial x_i$), f_{ij} the ij'th derivative ($\partial^2 f / \partial x_i \partial x_j$) and $|H|$ is the determinant of the Hessian matrix:

$$(3) \quad H = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{12} & f_{22} \end{bmatrix}$$

When more than two inputs are present in the technology there exists no 'empirical' formula for the elasticity of substitution. Chambers (1988) lists three alternatives, which are the most widely used and well known in the literature:

i. The direct elasticity of substitution (DES).
ii. The Allen Partial Elasticity of substitution (AES).
iii. The Morishima elasticity of substitution (MES).

The direct elasticity of substitution (DES) is defined by:

$$(4) \quad \sigma_{ij}^{DES} = \frac{(x_i f_i + x_j f_j)}{x_i x_j} \frac{f_i f_j}{(2 f_{ij} f_i f_j - f_{ii} f_j^2 - f_{jj} f_i^2)} \quad ; \quad i \neq j$$

i.e. Hick's two-input elasticity of substitution extended directly to the *n*-input case. This definition reduces to the two-input definition of the elasticity of substitution (equation 2) when $n=2$. As mentioned by Chambers (1988) this measure should be interpreted as a short-run elasticity, since it measures the degree of substitutability between factors i and j, keeping all other factors fixed. As such this measure has only limited use, as it does not allow for adjustment of all other factors when one factor is changed (see also Blackorby and Russel, 1989).

The Allen partial elasticity of substitution between two input factors x_i and x_j of the *n*-input production is defined by:

$$(5) \quad \sigma_{ij}^{A} = \frac{\sum_{k=1}^{n} x_k f_k}{x_i x_j} \frac{H_{ij}}{|H|} \quad ; \quad i \neq j$$

where $|H|$ is the determinant of the general Hessian matrix:

$$(6) \quad H = \begin{bmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{bmatrix}$$

and H_{ij} is the co-factor of the element f_{ij} in H. Comparison with equation (2) shows that when *n=2* the Allen partial elasticity reduces to the Hicks definition for two inputs. Blackorby and Russel (1989) criticise this measure for being inadequate even though it tries to remedy the drawbacks of the DES measure. They show that the AES measure does not preserve the properties of the original Hicks measure (equation 1), such as providing information on relative factor shares etc.

The Morishima elasticity of substitution (MES) is defined as:

$$(7) \quad \sigma_{ij}^{MES} = \frac{f_j}{x_i}\frac{H_{ij}}{|H|} - \frac{f_j}{x_j}\frac{H_{ij}}{|H|} \quad ; \quad i \neq j$$

Blackorby and Russel (1989) argue that this measure does preserve the properties of the original Hicks measure and should thus be preferred to the DES and the AES measures.

3. THE CES FUNCTION

An important question is which algebraic form a production function must possess in order to have constant elasticity of substitution between any two input factors. Arrow et al. (1961) have solved this problem for the two input case, i.e. for the Hicks elasticity of substitution measure, and the resulting function is the well-known Constant Elasticity of Substitution (CES) production function, the form of which is generally accepted as being *the* CES form.

Uzawa (1962), McFadden (1963) and Blackorby and Russel (1989) have derived the CES forms for the three different measures of *n*-input elasticity of substitution described above. The Blackorby-Russel form (constant Morishima elasticity of substitution) is a direct generalization to *n* inputs of the Arrow et al. two-input form, while the Uzawa and McFadden forms are more complicated but may be reduced to the Blackorby-Russel form as a special case. All three CES forms will be described in this section, but focus will in the remains of the paper be on the Blackorby-Russel CES function.

Arrow et al. (1961) show that a necessary and sufficient condition for the elasticity of substitution to be constant in the two-input case (equation 1 and 2) is that the production function has the form:

$$(8) \quad y = f_2(x_1, x_2) \equiv \begin{cases} \gamma(\beta \cdot x_1^{-\rho} + (1-\beta)x_2^{-\rho})^{-1/\rho} ; & \sigma \neq 1 \\ \gamma \cdot x_1^{\beta} \cdot x_2^{(1-\beta)} ; & \sigma = 1 \end{cases}$$

For this function the (constant) elasticity of substitution between the input factors is given by $\sigma = 1/(1+\rho)$, when *σ≠1*. Notice that the two input CES function reduces to the Cobb-Douglas form when *σ=1*.

McFadden (1963) shows, that a necessary and sufficient condition for an *n*-input production function to have constant elasticity of substitution under DES (equation 4) is that the function has the form:

$$(9) \quad \begin{aligned} &1 = \alpha_0 \sum_{s=1}^{S} \alpha_s \left[\Pi_{k \in N_s} \left(\frac{x_k}{y} \right) \right]^{-\rho} ; \rho \neq 0, \sum_{s=1}^{S} \alpha_s = 1 \\ &y = \alpha_0 \prod_{s=1}^{S} \left[\prod\nolimits_{k \in N_s} x_k \right]^{\alpha_s / m_s} \quad ; \quad \rho = 0 \end{aligned}$$

Where the *n* input-variables are partitioned into *S* distinct sets N_s (not necessarily with equal numbers of elements), for which $\bigcup_s N_s = (x_1, x_2, \ldots, x_n)$. Furthermore McFadden shows, that when the *S* partitions contain an equal number of elements *m*, equation (9) reduces to the more simple form:

$$(10) \quad y = \gamma \left[\sum_{s=1}^{S} \alpha_s \Pi_{k \in N_s} x_k^{-\rho} \right]^{-1/(\rho \cdot m)} ; \quad \rho < -\frac{1}{m} \quad , \quad \sum_{s=1}^{S} \alpha_s = 1$$

The constant DES elasticity of substitution of the McFadden form between any two inputs is given by

$$\sigma_{ij}^{DES} = 1/(1+\rho).$$

Uzawa (1962) shows, that a necessary and sufficient condition for an *n*-input production function to have constant elasticity of substitution given the AES measure of substitution (equation 5) is that the function has the form:

$$(11)\quad y = \Pi_{s=1}^{S}\left[\sum_{i\in N_s}\alpha_i x_i^{-\rho_s}\right]^{-\beta_s/\rho_s} \;;\quad \beta_s > 0,\quad \sum_{s=1}^{S}\beta_s = 1,\quad \alpha_i > 0,\quad -1 < \rho_s,\quad \rho_s \neq 0$$

with the same partition of the input set as in the DES case. The elasticity of substitution for this function is:

$$(12)\quad \sigma_{ij}^{AES} = \begin{cases} 1 \quad ; \quad i \in N_s, j \in N_t, s \neq t \\ \sigma_s = 1/(1+\rho_s) \quad ; \quad i, j \in N_s \end{cases}$$

Thus Uzawa relaxes the condition of constant elasticity of substitution between any two input factors to constant elasticity of substitution between two input factors from the same input set.

Finally Blackorby and Russel (1989) show that the MES measure of substitution (equation 7) is constant if and only if the production function has the form:

$$(13)\quad y = f(x_1,\ldots,x_k) = \begin{cases} \gamma\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right)^{-1/\rho} \quad ; \quad \rho \neq 1 \\ \gamma \cdot \prod_{k=1}^{n} x_k^{\beta_k} \quad ; \quad \rho = 1 \end{cases} \qquad \sum_{k=1}^{n}\beta_k = 1$$

As for the two-input case it is observed that this form reduces to the Cobb-Douglas form in the limit $\rho=0$. The constant elasticity of substitution between any two inputs under MES is $\sigma_{ij}^{MES} = 1/(1+\rho)$.

It is easily seen that the Uzawa form (11) reduces to equation (13) when all input values are contained in the same set, while the McFadden form (10) reduces to equation

(13) when the inputs are divided into *n* distinct sets each containing one input value. Blackorby and Russel (1989) argue, that the reason why the Uzawa and the McFadden forms are not directly comparable with the 'basic' CES structure given by Arrow et al. (equation 8) is that neither the DES nor the AES measures are natural generalizations of the two-variable elasticity of substitution. They continue by stating that since the MES measure gives a CES structure, which is directly comparable with the two-input case, the MES measure must be the 'true' measure of elasticity of substitution for multiple inputs.

As noted by Kmenta (1967A) the form given in equation (13) has returns to scale equal to unity, which is quite restrictive. A more general form, which allows the returns to scale to be different from unity, is (Kmenta, 1967A):

$$(14)\quad y = \gamma\left(\sum_{k=1}^{n} \beta_k x_k^{-\rho}\right)^{-\nu/\rho} \quad ; \qquad \sum_{k=1}^{n} \beta_k = 1$$

where ν is the return to scale. It is relatively straightforward to show that this function has constant elasticity of substitution given by $\sigma_{ij} = 1/(1+\rho)$ for all three measures of elasticity of substitution.

4. TRANSLOG APPROXIMATION OF THE TWO INPUT CES FUNCTION

This CES function is non-linear and cannot be linearised analytically as e.g. the Cobb-Douglas function (which is a special case of the CES function). Estimating functional parameters for the CES function thus includes non-linear fitting techniques, which are generally recognised as being complicated and to have convergence problems (local extrema etc.). But as Kmenta (1967A) notes, the two-input CES form may in certain cases be approximated by a linear translog form, which will be presented below.

For *n=2*, i.e. for two input factors, equation (14) reduces to the form:

$$(15)\quad y = f_2(x_1, x_2) \equiv \gamma(\beta \cdot x_1^{-\rho} + (1-\beta)x_2^{-\rho})^{-\nu/\rho}$$

Kmenta (1967A) states that the non-linear CES function given in (15) may in certain cases, depending on the magnitude of the parameter ρ, be approximated by a linear form, which may then be estimated by ordinary least squares techniques. This is possible when ρ is in the neighbourhood of zero (i.e. when the elasticity of substitution σ is in the neighbourhood of unity), as (15) can in this case be approximated with a Taylor series expansion around $\rho=0$. The resulting form of (15) becomes:

(16)

$$
\begin{aligned}
\ln(y) = \ln(\gamma) + \nu\cdot\beta\cdot\ln(x_1) + \nu\cdot(1-\beta)\cdot\ln(x_2) - \frac{\rho\nu}{2}\cdot\beta\cdot(1-\beta)\ln(x_1)^2 \\
-\frac{\rho\nu}{2}\cdot\beta\cdot(1-\beta)\ln(x_2)^2 + \rho\nu\cdot\beta(1-\beta)\ln(x_1)\cdot\ln(x_2) \\
= \ln(\gamma) + \nu\cdot\beta\cdot\ln(x_1) + \nu\cdot(1-\beta)\cdot\ln(x_2) - \frac{\rho\nu}{2}\cdot\beta\cdot(1-\beta)\left(\ln(x_1)-\ln(x_2)\right)^2
\end{aligned}
$$

The function given in (16) is recognized as a translog function:

(17)

$$
\ln(y) = \alpha_0 + \alpha_1\cdot\ln(x_1) + \alpha_2\cdot\ln(x_2) + \alpha_{11}\ln(x_1)^2 + \alpha_{22}\ln(x_2)^2 + \alpha_{12}\cdot\ln(x_1)\cdot\ln(x_2)
$$

for which the parameters fulfil the conditions:

(18)
$$
\begin{gathered}
\alpha_1 + \alpha_2 = \nu \\
\rho\cdot\nu\cdot\beta(1-\beta) = \alpha_{12} = -2\alpha_{11} = -2\alpha_{22}
\end{gathered}
$$

Equation (16) shows, that when $\rho=0$, i.e. when $\sigma=1$, the translog form reduces to the well-known Cobb-Douglas form, which by construction has $\sigma=1$. It will be shown in the appendix (for the more general *n*-input case), that while (16) is an approximation for $\rho>0$, the reduction to the Cobb-Douglas form is analytically exact in the limit $\rho=0$, i.e. that:

(19) $\lim_{\rho\to 0} f_2(x_1, x_2) = \gamma\cdot x_1^{\nu\beta}\cdot x_2^{\nu(1-\beta)}$

The above discussion shows, that if a two-input technology is believed (i) to have constant elasticity of substitution, and (ii) to have this elasticity of substitution in the neighbourhood of unity, then the input and output values observed for the technology may be fitted to the translog form (17), and the restrictions stated in equation (18) must then be utilized to (i) test whether the estimated translog form does in fact approximate a CES function, and (ii) estimate the CES parameters γ, ν, ρ and β.

It should in this context be emphasized that a given two-input technology *may* in fact be CES, even though the estimated translog form does *not* obey the restrictions given in (18). The reason may be that the necessary condition $\sigma \in O(1)$ is not true. Thus when the aim is to test whether a given technology fulfils CES conditions, it is *not* sufficient to fit data to a translog form and test if the conditions given in (18) are fulfilled. If the conditions are not fulfilled, and if it is still believed that the technology is in fact CES, the exact form given in equation (15) must be fitted to data employing non-linear techniques. A more throughout discussion of the ability of the translog approximation to estimate the true CES parameters is given in section 6.

5. TRANSLOG APPROXIMATION OF THE CES FUNCTION WITH *N* INPUTS

Kmenta's result presented in the preceding section is often cited (e.g. Campbell 1991, Pascoe and Robinson 1998), in the two-input as well as in the *n*-input case. But the result has, to the knowledge of the author of this paper, not to date been extended to the *n*-input case, and should thus not be cited when more than two inputs are present, as is often the case within e.g. fishery economics, where the purpose is frequently to investigate the mutual influence of several input factors on the landed catch of fish, employing different kinds of parametric production functions (e.g. Pascoe and Robinson 1998).

This section presents an extension of Kmenta's result to the *n*-input case, i.e. presents the Taylor approximation to the general *n*-input CES function. It is shown that the resulting approximation reduces to Kmenta's result for $n=2$, but that the restrictions on the translog parameters are in the *n*-input case more complicated than in Kmenta's two input case.

Thus, corresponding to the results presented in the previous section, i.e. that the two-input CES function (15) may for $\sigma \in O(1)$ be approximated by the translog function (17) given the restrictions on the parameters (18), it is shown in the appendix that when the general CES function (14) with n inputs is approximated by a first order Taylor approximation around $\rho=0$ ($\sigma \in O(1)$), the result is a translog function as in the two-input case, but that the restrictions on the translog parameters are not equal to the two-input restrictions given in (18). Hence it is shown in the appendix that:

The general CES function (14) may for $\sigma \in O(1)$ be approximated by the translog form:

$$(20)\quad \ln(y) = \ln(\gamma) + \sum_{i=1}^{n} \alpha_i \ln(x_i) + \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} \ln(x_i) \ln(x_j)$$

for which the parameters obey:

$$(21)\quad \begin{array}{ll} (I) & \alpha_k = \nu\beta_k \\ (II) & \sum_{k=1}^{n} \alpha_k = \nu \\ (III) & \left. \frac{\alpha_{ij}}{\alpha_i \alpha_j} \sum_{k=1}^{n} \alpha_k \right|_{i \neq j} = \frac{2\alpha_{ii}}{\frac{\alpha_i^2}{\sum_{k=1}^{n} \alpha_k} - \alpha_i} = \frac{2\alpha_{jj}}{\frac{\alpha_j^2}{\sum_{k=1}^{n} \alpha_k} - \alpha_j} = \rho \end{array}$$

Condition (III) may be rewritten to the following form:

$$(22)\quad \left. \alpha_{ij} \right|_{i \neq j} = \frac{-2\alpha_{ii}}{1 + \sum_{k \neq (i,j)} \frac{\alpha_k}{\alpha_j}} = \frac{-2\alpha_{jj}}{1 + \sum_{k \neq (i,j)} \frac{\alpha_k}{\alpha_i}}$$

which can be compared directly with the two-input case. Comparison of (22) and (18) shows that the *n*-input approximation reduced to Kmenta's result when $n=2$.

Thus if a *n*-input technology is believed to have constant elasticity of substitution, and if it is furthermore believed that this elasticity of substitution is in the neighbourhood of unity, then the input and output values observed for the technology may be fitted to the *n*-dimensional translog form (20), and the restrictions (III) stated in equation (21) must then be utilized to test whether the estimated translog form does in fact approximate a CES function. If this is the case, the three conditions (I), (II) and (III) can then be employed to estimate the CES parameters γ, ν, ρ and β.

It is clear, that the test of the restrictions (III) in (21) may be a lengthy and laborious procedure. It is in the appendix shown, that an alternative procedure may be followed, which requires a bit more manipulation of observed data, but also yields a more accessible testing procedures. I.e. it is shown in the appendix that:

The general CES function (14) may for $\sigma \in O(1)$ be approximated by the function:

$$(23)\quad \ln(y) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln(x_k) + \sum_{k=1}^{n-1} \sum_{m=k+1}^{n} \alpha_{km} \left(\ln(\frac{x_k}{x_m}) \right)^2$$

for which the parameters obey

$$(24)\quad \begin{aligned} &(i) && \alpha_k = \nu \beta_k \\ &(ii) && \sum_{k=1}^{n} \alpha_k = \nu \\ &(iii) && \frac{\alpha_{km}}{\alpha_k \alpha_m} = -\frac{\rho}{2\nu} \quad ; \quad \forall k,m \end{aligned}$$

This last identity (iii) in (24) shows that a necessary conditions for (23) to represent the Taylor approximation to the CES function is that the factors $\alpha_{km} / (\alpha_k \alpha_m)$ are equal to the same constant for all *k,m*, i.e. especially that the factors are equal for all *k,m*. This is quit an easier condition to test than the conditions (III) in (21) for the direct translog approximation.

Thus an alternative to employing the direct translog function (20) to test whether an observed dataset complies with a CES function is firstly to calculate the identities $\ln(x_k/x_m)^2$ (or alternatively $(\ln(x_k) - \ln(x_m))^2$) for all *k,m*, together with the identities $\ln(y)$ and $\ln(x_k)$ for all *k*. This gives a bit more introductionary labour, but this is by far regained by the testing procedures (iii) in (24) when compared to (III) in (21). If the identity (iii) in (24) is fulfilled, and if the fit of the function (23) to data is generally acceptable, it may be concluded that the CES function is a good fit to data, and the translog parameters may then be estimated from conditions (i), (ii) and (iii) in (24).

It must as in the previous section be stressed, that the technology considered *may* in fact be CES, even though the estimated linear forms (20) or (23) does not comply with the restrictions given in (21) respectively (24). The ability of the translog approximation to predict the *n*-input CES function is discussed in the next section.

6. CONSISTENCY OF THE TRANSLOG APPROXIMATION TO THE CES FUNCTION

Thursby and Knox Lovell (1978) discuss the consistency of the CES parameters estimated with Kmenta's translog approximation (16) to the two-input CES function (15). They point to the fact that the translog approximation is a truncated Taylor series and that the hereby-estimated CES parameters must thus necessarily be biased by a truncation error. They moreover stress that the estimated CES parameters may not even be asymptotically consistent when the examined sample size increases, as the Taylor series underlying the approximation is a power series in $\ln(x_1/x_2)$, which as such only converge to the true CES function when $\ln(x_1/x_2)$ is within the convergence circle having radius $|1/(\rho\beta)|$. Thursby and Knox Lovell perform a series of Monte Carlo simulations to test how well the two-input approximation estimates the CES parameters and generally conclude that 'the CES parameters are estimated consistently only under the most favourable circumstances' (Thursby and Knox Lovell, 1978), i.e. for the input values and CES parameters limited within very restricted regimes.

The result of Thursby and Knox Lovell logically extends to the *n*-input case, meaning (i) that the CES parameters estimated by the translog approximation are expected to be biased by the cut-off of the Taylor approximation, and (ii) that the Taylor series underlying the approximation will be a power series in the $(n-1)$ terms $\ln(x_i/x_1)$ whose convergence radii will also depend on the (unknown) CES parameters, again resulting in a violation of asymptotically consistency of the estimated CES parameters if the input variables do not comply with the convergence radii.

No attempt has been made in the present work to determine the convergence radii analytically in the *n*-input case. However the truncation error as well as bias and consistency of the *n*-input approximation have been tested statistically, the results of which are presented below.

6.1. Effect of CES parameter, input-range and input order of magnitude

Two examples have been constructed to estimate the magnitude of the truncation error of the translog approximation to the CES function (i) when the CES parameter are varied and (ii) when of the range respectively order of magnitude of the input values are varied. The two examples have three respectively five input parameters. Table 1 shows the CES parameters of the examples.

Table 1. Parameters of test examples. A bracket {} indicates that the approximation to the CES function has been tested for each parameter within the bracket. Notation as in equation (14)

Parameter	Example I: 3 inputs	Example II : 5 inputs
β_1	0.35	0.4
β_2	0.4	0.3
β_3	0.25	0.1
β_4	-	0.1
β_5	-	0.1
γ	{5,50}	{5,50}
ν	{0.5,2,10}	{0.5,2,10}
ρ	{0.1 , 0.2,..., 1}	{0.1 , 0.2,..., 1}

Thus in order to estimate the truncation error as a function of the CES parameters, the translog approximation has in each case been tested for two values of the scale parameter γ (5 and 50), three values of the returns to scale ν (0.5, 2 and 10) and 10 values of the parameter ρ (ranging from 0.1 to 1 with step 0.1). $\gamma=5$ and $\nu=2$ is defined as the basis scenario against which the remaining scenarios are compared. I.e. either γ is kept constant at the value *5* while ν is varied or vice versa.

Likewise in order to estimate the truncation error as a function of range and order of magnitude of input values, values ranging from 10 to 10^5 have been employed. The error has been estimated as a function of input range by employing input values in intervals of increasing length given by:

- **R1)** 10 to 110 with step 20, i.e. inputs taking the values (10, 30, 50, 70, 90, 110).
- **R2)** 10 to 1010 with step 200, i.e. inputs taking the values (10, 210, 410, 610, 810, 1010).
- **R3)** 10 to 10010 with step 2000, i.e. inputs taking the values (10, 2010, 4010, 6010, 8010, 10010).
- **R4)** 10 to 100010 with step 20000, i.e. inputs taking the values (10, 20010, 40010, 60010, 80010, 100010).

The truncation error has been estimated as a function of order of magnitude of inputs by employing input values coming from intervals confined to 10, 10^2, 10^3 and 10^4. More specifically input values for the simulations have been taken from four different intervals defined as:

M1) 10 to 110 with step 20, i.e. order of magnitude $10\text{-}10^2$ (each input can take the values (10, 30, 50, 70, 90, 110)).

M2) 110 to 1110 with step 200, i.e. order of magnitude $10^2\text{-}10^3$ (each input can take the values (110, 310, 510, 710, 910, 1110)).

M3) 1010 to 11010 with step 2000, i.e. order of magnitude $10^3\text{-}10^4$ (each input can take the values (1010, 3010, 5010, 7010, 9010, 11010)).

M4) 10010 to 110010 with step 20000, i.e. order of magnitude $10^4\text{-}10^5$ (each input can take the values (10010, 30010, 50010, 70010, 90010, 110010)).

The truncation error of employing the Taylor approximation to the CES function has been measured as:

$$(25)\quad \Delta \equiv \frac{f^{CES}_{exact}(x_1,\ldots,x_n) - f^{CES}_{trans\log}(x_1,\ldots,x_n)}{f^{CES}_{exact}(x_1,\ldots,x_n)}$$

where f^{CES}_{exact} is the exact CES function (14) and $f^{CES}_{trans\log}$ the approximative CES function (20) with the restrictions (21) employed to calculate the translog parameters.

For each of the parameter and input scenarios the relative truncation error (25) has been evaluated for each distinct input combination, and the maximum Δ_{max} and minimum Δ_{min} of these relative errors recorded. E.g. in the 5-input case, where the inputs are ranging between 10 and 110 with step 20, each of the five inputs can realize the six values (10, 30, 50, 70, 90, 110), leading to 6^5=7776 input combinations. The relative error (25) is calculated for each of these input sets, and finally the maximum and minimum values of this set of 7776 relative errors are recorded. Δ_{max} and Δ_{min} as such give the total range of the truncation error of the translog approximation.

Figure 1-8 show the results of testing the influence of the range of the input parameters. Each figure shows Δ_{max} and Δ_{min} as a function of ρ for each of the four different input ranges R1 to R4, given a specific (γ,ν) scenario. E.g. figure 1 shows the Δ_{max} and Δ_{min} for the basic scenario $\gamma=5$ and $\nu=2$ in the three-input case. In the figure the 'R1 Min' and 'R1 Max' curves shows the Δ_{min} and Δ_{max} values for input range R1 (cf. page 15) and for each test value of ρ, and likewise for the remaining curves.

Figure 1 and 2 show the basic scenario $\gamma=5$ and $\nu=2$ for three respectively five inputs (i.e. for example I respectively II in table 1). Comparison of the two figures firstly shows that the truncation error of the translog approximation increases with increasing number of input variables, when the input range is more than one order of magnitude (R2 to R4). This observation is confirmed by figure 3 to 8.

The two figures secondly show what must be expected, that the truncation error of the translog approximation to the CES function increases when ρ increases, for three as well as for five input values. This is confirmed by figure 3 to 8 as well.

Thirdly the two figures show that the truncation error of the translog approximation increases when the range of the input values increases for three as well as for five inputs. This is expected given the results presented by Thursby and Knox Lovell (1978), as the parameters $\ln(x_i/x_1)$ must at some point exceed the convergence radii as the range of the input values increases. The two figures generally indicate that the range of the input values should not be more than one order of magnitude as the truncation error of the approximation decreases fast for the other three range regimes R2 to R4 (cf. page 15). This observation is confirmed by figure 3 to 8.

Figure 3 and 4 show the relative truncation errors for three respectively five inputs when $\gamma=50$ is employed. Comparison with figure 1 and 2 (the basic scenario) indicates, that increasing the γ value does not change the accuracy of the approximation. Further tests have shown that the value of γ does not generally affect the magnitude of the truncation error.

Figure 5 and 6 show the relative errors when $\nu=10$ for three respectively five inputs, while figure 7 and 8 show the relative errors for $\nu=0.5$. Comparison with figure 1 and 2 indicates that the accuracy of the translog approximation decreases with increasing ν, i.e. with increasing returns to scale.

When the influence of the order of magnitude of the input values (M1 to M4) is tested, it turns out that the order of magnitude does not influence the accuracy of the approximation. I.e. that the accuracy of the translog approximation to the CES function is not changed by moving one or more orders of magnitude up or down, independently of the magnitude of the remaining parameters γ, ν and ρ.

To summarise, the above-described estimates of the truncation error of the translog approximation to the CES function have indicated the following guidelines for applicability of the approximation:

- The substitution parameter ρ should not exceed 0.1-0.2, i.e. the elasticity of substitution σ should not be much less than 0.9.
- Decreasing returns to scale ($\nu<1$) should generally be preferred.
- The range of the input variables should generally be at the most one order of magnitude.

Thus generally a translog approximation to the CES function must be handled with care, and avoided if it is suspected (i) that the elasticity of substitution is much less than 0.9 and/or (ii) that there is increasing returns to scale, and/or (iii) the observed input values can not be scaled to be within at the most one order of magnitude of each other.

6.2. Bias and consistency

The bias and consistency of the CES parameters estimated by the translog approximation have been tested against variation of the dependent variable y and number N of observations included in the estimation.

A dataset has been constructed for which the dependent variable y is calculated from the vector of explanatory variables $\boldsymbol{x}$ with the CES function (14). This dataset is thus an exact fit to the true CES function. The input vector has been chosen to be three-dimensional (x_1, x_2, x_3) and confined to the interval $[1;106]$. The CES parameters of the example have been set to (notation as in equation 14):

- $\beta_1=0.35$
- $\beta_2=0.4$
- $\beta_3=0.25$
- $\gamma=5$
- $\nu=0.5$
- $\rho=0.1$.

The choice of parameters and range of input values ensures that the constructed example is within safe truncation limits as outlined in section 6.1.

It is additionally assumed that the logarithm of the observed dependent variable y is iid. normally distributed around the true ln(y) with standard deviation s_0. Thus for a given set of input values (x_1, x_2, x_3) the logarithm of the output value is determined by:

$$(26)\quad \ln(y) = \ln\left[5\left(0.35 \cdot x_1^{-0.1} + 0.4 \cdot x_2^{-0.1} + 0.25 \cdot x_3^{-0.1}\right)^{-0.5/0.1}\right] + u$$

where u is iid N($0,s_0$). Standard deviations of 0, 0.01, 0.1, 0.3 and 0.5 have been tested.

Four different sets of input values in the range [*1;106*] have been tested with varying number N of observations. The four sets are:

i) Inputs in the range 1 to 101 with step 50, resulting[3] in $N=27$.

ii) Inputs in the range 1 to 101 with step 25, resulting in $N=125$.

iii) Inputs in the range 1 to 101 with step 20, resulting in $N=216$.

iv) Inputs in the range 1 to 106 with step 15, resulting in $N=512$.

For each of these sets of input values, 1000 realisations have been created of the dataset $\left(x_{i,1} x_{i,2} x_{i,3}, \ln(y_i)\right)$, where $x_{i,j}$ is the j'th input for the i'th observation (i running from *1* to N), and ln(y_i) the logarithm of the i'th dependent variable, calculated from equation (26) using a random number generator for the normal distribution $N(0, s_0)$.

Each constructed dataset thus has CES properties, i.e. will comply with the CES function (14), the fit of which to the dataset will only depend on the stochastic error s_0.

Finally the function (23) is fitted to each realisation of the dataset (using Ordinary Least Square regression) and the conditions (24) have been employed in each run (a) to test whether the fitted linear function (23) predicts the CES structure of data (employing condition (iii) in equation 24) and (b) to obtain estimates of the CES parameters γ, ν and ρ.

[3] Each of the three inputs can take the values (1 , 51 , 101), and thus there is in total $N=3^3=27$ different input combinations.

Thus for each value of N and s_0 1000 fits to equation (23) and 1000 corresponding tests of equation (24) have been obtained, resulting in distributions of the regression coefficient of determination R^2, of the test parameter for condition (iii) in (24) and of the CES parameters γ, ν and ρ estimated by (24). The following characteristics of these distributions have been recorded for each N and s_0:

- The 25%, 50% (median) and 75% quantiles of the resulting R^2 distributions.

- The percentage of experiments for which the test (iii) in (24) is rejected on a 5% level. The null hypothesis of the test is that the fractions $\alpha_{km}/(\alpha_k \alpha_m)$ are equal, and thus the recorded percentage indicates the overall probability of falsely rejecting the true CES structure of data.

- 25%, 50% (median) and 75% quantiles for the γ, ν and ρ distributions for the sample of experiments for which the null hypothesis has been accepted. In each of these experiments the ρ values have been calculated as the arithmetic mean of the three ρ values obtained from condition (iii) in (24).

The recorded R^2 values estimate the truncation error of the translog approximation as a function of variation of the dependent variable and as a function of number of observations. The probability of rejection and the γ, ν and ρ distributions estimates the consistency of the translog approximation.

Table 2-5 presents the results of the simulations. The four tables firstly show that the R^2 value is generally high when $s_0=0$, indicating a very good fit of the CES data to the translog function when no stochastic variation is included.

It is moreover observed from the tables that the ρ and γ values obtained for $s_0=0$ are biased some (the true value are *0.1* and *5*), which is due to the truncation of the Taylor approximation discussed above.

Finally it is seen that the null hypothesis of CES properties of data is accepted for $s_0=0$ in the cases $N=27$ and for $N=125$, while it is rejected for $N=216$ and for $N=512$. The null hypothesis thus shifts from being accepted to being rejected for N somewhere around 200 observations when there is no stochastic variation of the dependent variable. This is expected, as the deviation of the estimated translog parameters decreases as the number of observations N increases, thus making the test (24) more restrictive. This indicates the rather important result that the translog approximation to the CES function will not in general asymptotically predict true CES structure, due to the truncation error of the Taylor approximation.

The four tables secondly show that the overall fit R^2 of equation (23) to data decreases with increasing N and with increasing s_0 when $s_0>0$. That R^2 decreases with increasing N when $s_0>0$ is assumed to be due to the fact that the errors of each observation are amplified over the number of observations in the fit, and thus the overall fit of (23) decreases for increasing number of observations. That R^2 decreases with increasing s_0 is expected, as the fit will naturally be less reliable when the error of the dependent variable increases.

The four tables further show that when $s_0>0$ there is less than a 10% chance of neglecting the null hypothesis of CES structure, and that this probability seems to decrease with increasing s_0. This is expected due to the fact that the error on the regression parameters of equation (23) will increase with increasing s_0 and thus condition (iii) in (24) becomes less restrictive. On the other hand there does not seem to be any relationship between number of observations N and the probability for neglecting the null hypothesis.

Finally the four tables show that the range of the CES parameters estimated when the null hypothesis is accepted increases with increasing s_0 and decreases with increasing N, and that this effect is most severe for the substitution parameter ρ and the scale γ, while it is not so pronounced for the returns to scale ν. This result is in accordance with what is found by Thursby and Knox Lovell (1978). The effect is expected as the error on the estimated CES parameters will naturally increase with increasing error on the dependent variable, but decrease with increased number of observations, as the latter will reduce the error on the regression parameters.

Thus generally these results indicate, that if data does actually possess CES structure, and if the dependent variable does have some stochastic variation, then the translog approximation to the exact CES function will predict this CES structure in more than 9 out of 10 cases.

On the other hand the consistency of the hereby estimated CES parameters will depend strongly on the variation of the dependent variable and moreover on the number of observations.

It is finally also observed that the less the variation is of the dependent variable the greater will the chance be for neglecting the null hypothesis even though it is true, due to the truncation error of the Taylor function.

The conclusion is that *if* the null hypothesis is accepted, care should be taken in trying to determine the error of the thus estimated CES parameters, while if the null is *not* accepted the function may still *be* a CES but just with small variation of the dependent variables.

The latter observation leads back to the earlier discussion, that if the translog approximation of the CES function does *not* predict CES properties of the dataset in question, this is actually not a final indication that data does not have CES properties, as the translog approximation test is only necessary but certainly not sufficient. If the null hypothesis is therefore neglected, effort must be made in performing a non-linear fit of data to the exact CES function (14).

7. DISCUSSION AND CONCLUSION

This paper presents a generalization to *n* inputs of Kmenta's (1967A) translog linearisation of the two-input CES function. It is shown that the *n*-input CES function may generally be approximated by a translog function employing a first order truncated Taylor series, when the elasticity of substitution is in the neighbourhood of unity. It is moreover shown, that while quit simple restrictions are shown to hold on the translog parameters in the two-input case (cf. Kmenta, 1967A), these restrictions become more complicated in the *n*-input case, but that the *n*-input result reduces to Kmenta's result for $n=2$. An alternative testing procedure is presented, which is a re-writing of the translog function and thus still linear, but is more simple to employ in the test of CES properties than the direct translog function.

The truncation error together with bias and consistency of employing the translog approximation are moreover discussed. It is shown that the translog approximation is only valid for a limited range of the elasticity of substitution, and that the translog approximation should be employed only if decreasing returns to scale is expected, and if the input values of the problem can be scaled to have a variation of at the most one order of magnitude.

Finally it has been shown that the ability of the translog approximation to predict true CES structure is generally high when there is some stochastic variation of the observed dependent variable (which may often be expected), and that the error of the estimated CES parameters generally increases with increasing variation of the dependent variable but decreases with increasing number of observations.

As a conclusion it must be stressed that a given *n*-input technology *may* in fact be CES, even though the estimated translog form does *not* predict CES structure of data. The reason may be, that the necessary condition $\sigma \in O(1)$ is not true, or that one or more of the other conditions listed above are not met. I.e. the CES test for the translog function is a necessary but by no means sufficient condition for CES properties of data.

Thus when the aim is to test whether a given technology fulfils CES conditions, it is *not* sufficient to fit data to a translog form and test whether this function predicts translog properties of data. If the test is neglected the exact CES form given (14) should be fitted to data.

Table 2. Test of the influence of stochastic variation of the dependent variable on the bias and consistency of the translog approximation to the CES function with N=27 observations in the dataset. 'Q1'=25% quantile, 'Q2'=50% quantile (median) and 'Q3'=75% quantile of distributions.

N=27		s_0: 0	0.01	0.1	0.3	0.5
R^2	Q1		0.999742	0.977393	0.830583	0.645861
	Q2	0.999977	0.999792	0.981777	0.863878	0.710382
	Q3		0.999892	0.985313	0.890224	0.767077
Reject		0%	6.1%	7.8%	7.1%	4.1%
ρ	Q1		0.1006843	0.07199213	0.0134944	-0.0601043
	Q2	0.103877	0.1041359	0.10603599	0.1207598	0.1456164
	Q3		0.1072417	0.14182115	0.2349434	0.3883328
ν	Q1		0.496758	0.485271	0.457711	0.435039
	Q2	0.49811	0.498128	0.498337	0.495067	0.495701
	Q3		0.499518	0.511711	0.535462	0.559876
γ	Q1		5.00819	4.76531	4.38303	3.86613
	Q2	5.03523	5.03553	5.02138	5.09524	5.10422
	Q3		0.06372	5.33428	5.94481	6.71214

Table 3. Test of the influence of stochastic variation of the dependent variable on the bias and consistency of the translog approximation to the CES function with N=125 observations in the dataset. 'Q1'=25% quantile, 'Q2'=50% quantile (median) and 'Q3'=75% quantile of distributions.

N=125		s_0: 0	0.01	0.1	0.3	0.5
R^2	Q1		0.999618	0.964087	0.747466	0.515773
	Q2	0.999987	0.999649	0.967022	0.769215	0.551196
	Q3		0.999679	0.969823	0.786951	0.587799
Reject		0%	6.3%	4.6%	5.9%	4.7%
ρ	Q1		0.1030322	0.0827412	0.05111057	0.0147659
	Q2	0.10546	0.1052476	0.1054818	0.11464814	0.1300330
	Q3		0.1074796	0.1283289	0.18900186	0.2620772
ν	Q1		0.497259	0.507229	0.469041	0.453258
	Q2	0.498145	0.498095	0.497877	0.496429	0.498362
	Q3		0.499152	0.489295	0.524019	0.538532
γ	Q1		5.01680	4.85345	4.50341	4.24482
	Q2	5.03712	5.03710	5.03632	5.07871	5.01430
	Q3		5.05500	5.22385	5.69794	6.02682

Table 4. Test of the influence of stochastic variation of the dependent variable on the bias and consistency of the translog approximation to the CES function with N=216 observations in the dataset. 'Q1'=25% quantile, 'Q2'=50% quantile (median) and 'Q3'=75% quantile of distributions.

N=216		s_0				
		0	0.01	0.1	0.3	0.5
R^2	Q1		0.999565	0.959637	0.722376	0.480528
	Q2	0.999989	0.999593	0.961958	0.738024	0.506749
	Q3		0.999620	0.964366	0.754917	0.537153
Reject		100%	5.2%	5.4%	4.9%	6.1%
ρ	Q1		0.1038736	0.0857869	0.04802916	0.0377255
	Q2	0.105467	0.1056101	0.1065727	0.10508421	0.1284052
	Q3		0.1075181	0.1246820	0.16845500	0.2352758
ν	Q1		0.497462	0.490426	0.477696	0.454299
	Q2	0.498292	0.498235	0.498375	0.501457	0.495497
	Q3		0.499039	0.506229	0.523193	0.534070
γ	Q1		5.01978	4.88316	4.54437	4.33639
	Q2	5.03433	5.03561	5.03168	4.94939	5.06866
	Q3		5.05085	5.19298	5.49269	6.00490

Table 5. Test of the influence of stochastic variation of the dependent variable on the bias and consistency of the translog approximation to the CES function with N=512 observations in the dataset. 'Q1'=25% quantile, 'Q2'=50% quantile (median) and 'Q3'=75% quantile of distributions.

N=512		s_0				
		0	0.01	0.1	0.3	0.5
R^2	Q1		0.999488	0.951863	0.686396	0.435922
	Q2	0.999992	0.999509	0.953772	0.699865	0.455359
	Q3		0.999529	0.955914	0.711085	0.474413
Reject		100%	7.3%	5.2%	5.9%	3.9%
ρ	Q1		0.1039587	0.0922761	0.0703005	0.04504422
	Q2	0.105335	0.1052505	0.1052290	0.1084335	0.11370608
	Q3		0.1065867	0.1185540	0.1478851	0.19390381
ν	Q1		0.497934	0.493190	0.481723	0.468192
	Q2	0.498534	0.498531	0.498635	0.499289	0.496859
	Q3		0.499129	0.504441	0.516409	0.526893
γ	Q1		5.01757	4.91110	4.66048	4.48144
	Q2	5.02995	5.02958	5.03099	5.02116	5.07769
	Q3		5.04208	5.14001	5.37464	5.68456

Figure 1. Test of input range for three inputs, basic scenario γ=5 and ν=2. ‘R1’ to ‘R4’ are input range regimes and ‘Min’ resp. ‘Max’ are min resp. max error of approximation.

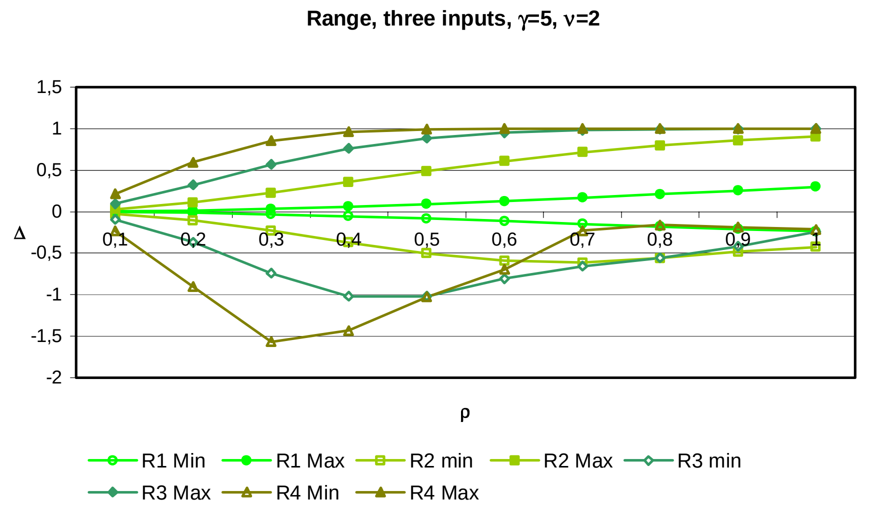


Figure 2. Test of input range for five inputs, basic scenario γ=5 and ν=2. ‘R1’ to ‘R4’ are input range regimes and ‘Min’ resp. ‘Max’ are min resp. max error of approximation.

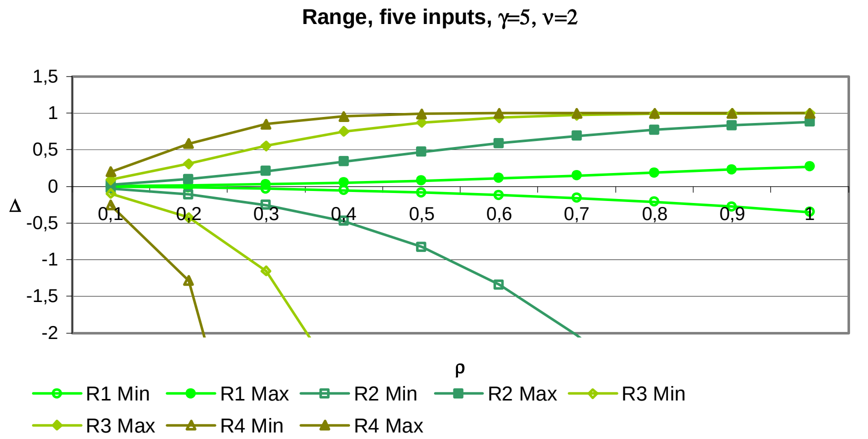


Figure 3. Test of input range for three inputs, γ=50 and ν=2. 'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.

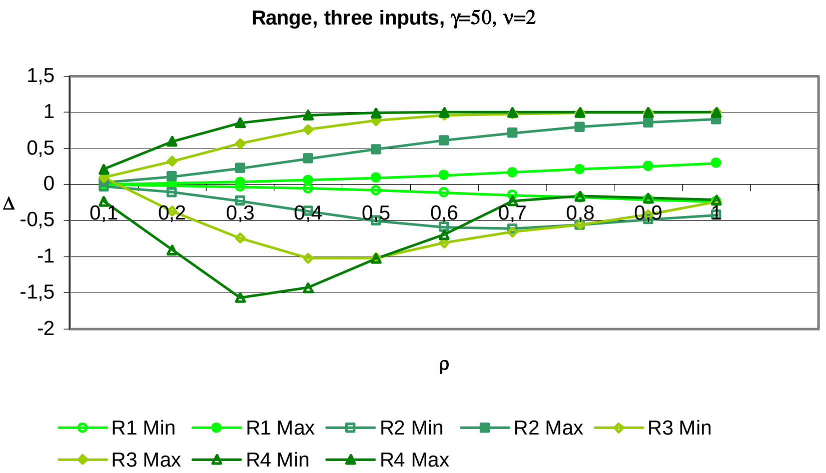


Figure 4. Test of input range for five inputs, γ=50 and ν=2. 'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.

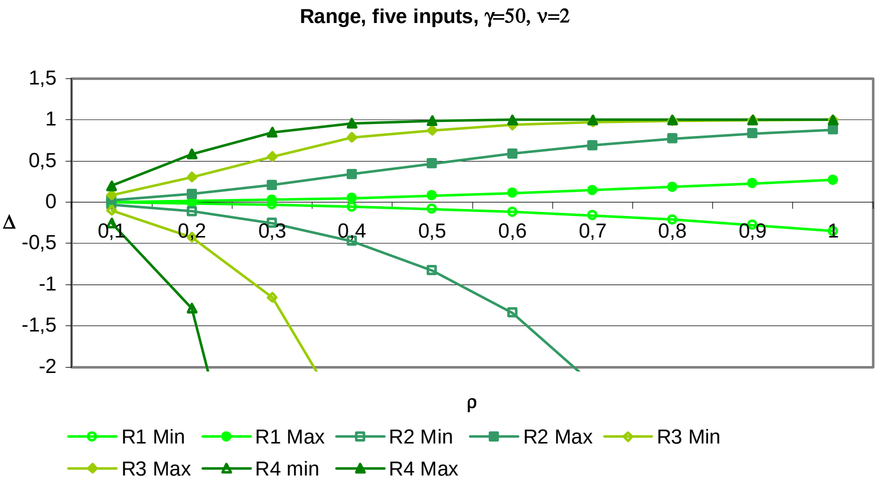


Figure 5. Test of input range for three inputs, γ=5 and ν=10. 'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.

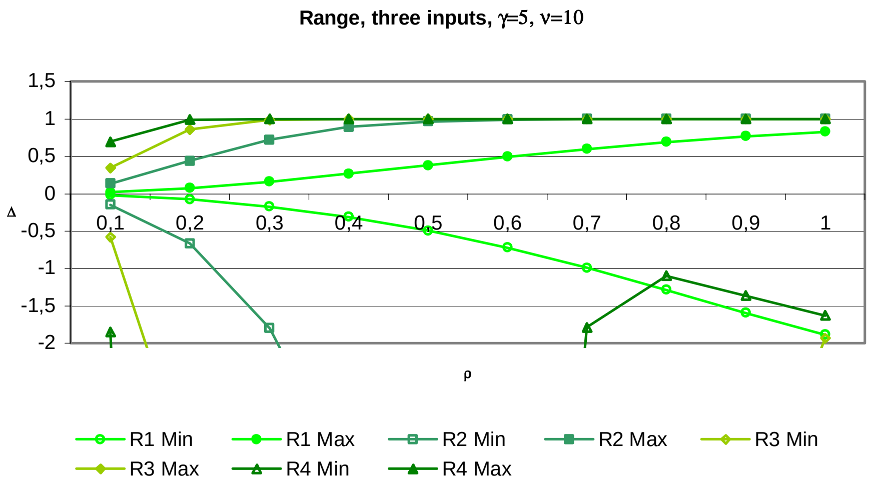
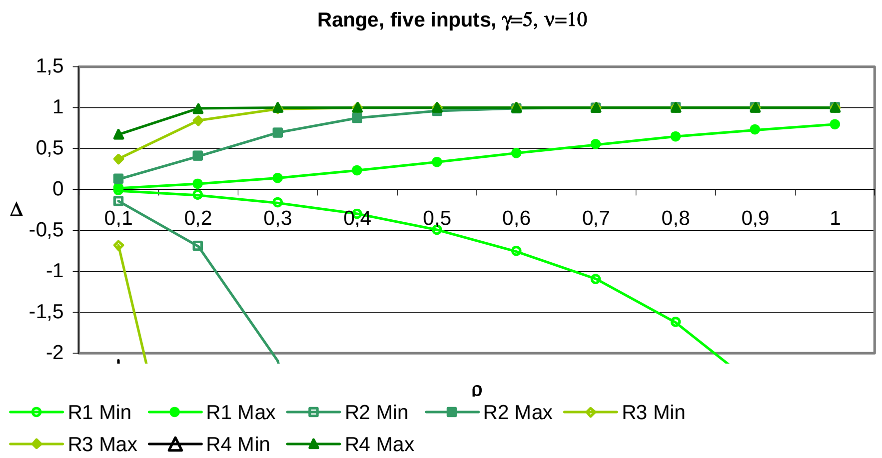


Figure 6. Test of input range for five inputs, γ=5 and ν=10. 'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.



Note: R4 min. Are out of graph range

Figure 7. Test of input range for three inputs, γ=5 and ν=0.5. 'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.

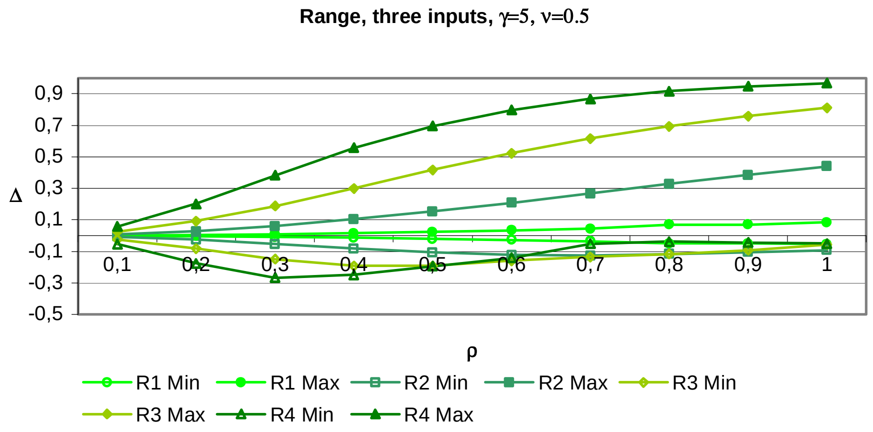
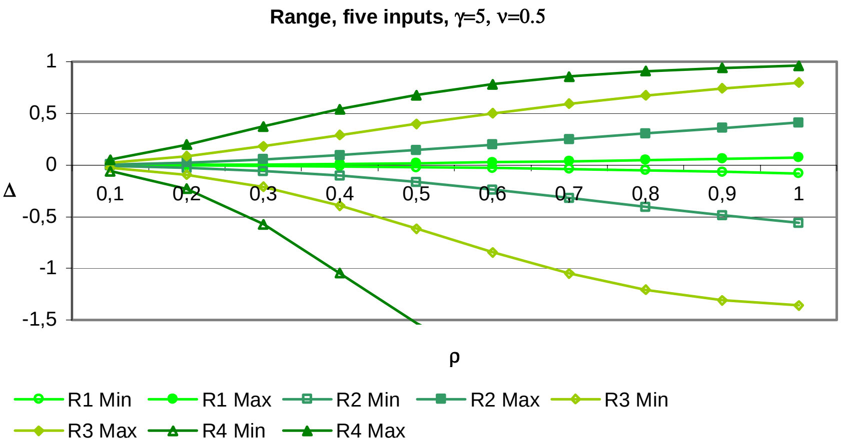


Figure 8. Test of input range for five inputs, γ=5 and ν=0.5.'R1' to 'R4' are input range regimes and 'Min' resp. 'Max' are min resp. max error of approximation.



Appendix A

The proof of equation (20) and (21) builds on ***Taylor's formula***:

If $f\colon I \rightarrow \Re$ is n times differentiable on $I \subset \Re$ and if $f^{(n)}$ (the n'th differential of f) is continuous in $a \in I$, then

$$(27)\quad f(x) = f(a) + \sum_{i=1}^{n} \frac{1}{i!} f^{(i)}(a)(x-a)^i + O\left((x-a)^{i+1}\right)$$

The proof of this is part of any introductionary text on mathematical analysis. It should especially be noticed that f must be continuous in a for the formula to hold, i.e. have a well-defined limiting value in $x=a$. This formula will be employed for $n=1$ to prove equation (20) and (21). Moreover ***L'Hospital's rule*** will be employed:

If $f,g\colon I \rightarrow \Re$ both are n times continuously differentiable on $I \subset \Re$ and if:

$$(28)\quad \begin{aligned} f(a) &= f^{(1)}(a) = \cdots = f^{(n-1)}(a) = 0 \\ g(a) &= g^{(1)}(a) = \cdots = g^{(n-1)}(a) = 0 \end{aligned}$$

for $a \in I$, and if further $g^{(n)}(a) \neq 0$, then $f(x)/g(x)$ is defined in a and

$$(29)\quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

Taylor's formula given by equation (27) will be employed directly in the proof of equation (20) and (21). It may be argued that it is simpler to use the following lemma (the proof of which is straightforward):

Lemma 1
Let $f \in C^n(\Re)$ be given by

$$(30)\quad f(x) = \frac{g(x)}{x}$$

Where $g \in C^{n+1}(\Re)$. Assume furthermore that $f(0) = \lim_{x \rightarrow 0} g(x)/x = a$, i.e. that f is defined in $x=0$ with the value a. Then the Taylor polynomial of the n'th order of f in $x=0$, $f_T^n(x)$ is given by:

(31) $$f_T^n(x) = a + \sum_{i=1}^{n} \frac{1}{i!} f^{(i)}(0)x^i = \frac{1}{x}\sum_{i=1}^{n+1} \frac{1}{i!} g^{(i)}(0)x^i = \frac{1}{x} g_T^{n+1}(x) \quad ; \quad x \neq 0$$

The disadvantage of employing the lemma is that the Taylor approximation of *f* ends up being defined on $\Re/\{0\}$. Furthermore the lemma does not encourage the user to investigate whether *f* is actually *defined* in zero or not, which is a basic condition for the Taylor approximation to hold. It is thus more 'sound' to employ Taylor's formula directly for *f*, although this involves a bit more labour.

Thus Taylor's formula (27) will be employed for $n=1$ to approximate the CES function given in equation (14). This function may be rewritten to:

(32) $$\frac{1}{\nu}\left(\ln(y) - \ln(\gamma)\right) = -\frac{1}{\rho}\ln\left(\sum_{k=1}^{n} \beta_k x_k^{-\rho}\right)$$

Define

(33) $$f(\rho) = \frac{1}{\rho}\ln\left(\sum_{k=1}^{n} \beta_k x_k^{-\rho}\right)$$

It is this function *f* which will be Taylor approximated to the first order around $\rho=0$. The first step is to determine $f(0)$:

$$f(0) = \lim_{\rho\to 0} f(\rho) = \lim_{\rho\to 0} \frac{\ln\left(\sum_{k=1}^{n} \beta_k x_k^{-\rho}\right)}{\rho}$$

$$= \lim_{\rho\to 0} \frac{\frac{\partial}{\partial\rho}\ln\left(\sum_{k=1}^{n} \beta_k x_k^{-\rho}\right)}{\frac{\partial}{\partial\rho}(\rho)}$$

(34) $$\Leftrightarrow f(0) = \lim_{\rho\to 0} f(\rho) = \lim_{\rho\to 0} \frac{\left(\frac{-\sum_{k=1}^{n} \beta_k x_k^{-\rho}\ln(x_k)}{\sum_{k=1}^{n} \beta_k x_k^{-\rho}}\right)}{1} = -\sum_{k=1}^{n} \beta_k \ln(x_k)$$

where it is used (i) that the numerator as well as denominator of f are equal to zero for $\rho=0$, and thus that L'Hospital's rule (27) may be applied, and (ii) that the sum of the β values is equal to unity. This shows that f is continuous in $\rho=0$, and thus that the Taylor approximation of f may be applied around $\rho=0$.

The first derivative of f is:

$$\frac{\partial}{\partial\rho} f = -\frac{1}{\rho^2}\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right) + \frac{1}{\rho}\frac{-\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)}{\sum_{k=1}^{n}\beta_k x_k^{-\rho}}$$

(35) $$\frac{\partial}{\partial\rho} f = \frac{-\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right)\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right) - \rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)}{\rho^2\sum_{k=1}^{n}\beta_k x_k^{-\rho}} \equiv \frac{N(\rho)}{D(\rho)}$$

It is easily seen that the numerator N as well as denominator D of this expression are zero for $\rho=0$, and thus L'Hospital's rule may be applied to find the value of the derivative of f in $\rho=0$. For this purpose the derivatives of N and D must be determined:

(36) $$\begin{aligned}\frac{\partial}{\partial\rho}N &= \left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)\right)\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right) + \left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right)\frac{\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)}{\sum_{k=1}^{n}\beta_k x_k^{-\rho}} - \\ &\quad \sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k) + \rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)^2 \\ &= \left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)\right)\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right) + \rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)^2\end{aligned}$$

and:

(37) $$\frac{\partial}{\partial\rho}D = 2\rho\sum_{k=1}^{n}\beta_k x_k^{-\rho} - \rho^2\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)$$

Thus

$$\lim_{\rho\to 0}\frac{\partial}{\partial\rho}f=\lim_{\rho\to 0}\frac{\frac{\partial}{\partial\rho}N}{\frac{\partial}{\partial\rho}D}=\lim_{\rho\to 0}\frac{\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)\right)\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right)+\rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)^2}{2\rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}-\rho^2\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)}$$

$$=\lim_{\rho\to 0}\frac{\frac{1}{\rho}\ln\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\right)\left(\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)\right)+\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)^2}{2\sum_{k=1}^{n}\beta_k x_k^{-\rho}-\rho\sum_{k=1}^{n}\beta_k x_k^{-\rho}\ln(x_k)}$$

$\Rightarrow$

$$(38)\ \lim_{\rho\to 0}\frac{\partial}{\partial\rho}f=\frac{-\sum_{k=1}^{n}\beta_k\ln(x_k)\sum_{m=1}^{n}\beta_m\ln(x_m)+\sum_{k=1}^{n}\beta_k\ln(x_k)^2}{2}$$

It has been employed that the sum of the β values is equal to unity, and that the limiting value of $\ln(\sum_{k=1}^{n}\beta_k x_k^{-\rho})/\rho$ is equal to $-\sum_{k=1}^{n}\beta_k\ln(x_k)$, as shown in equation (34).

The first order Taylor approximation of *f* thus becomes:

$$f(\rho)=f(0)+\rho\frac{\partial}{\partial\rho}f(\rho)|_{\rho=0}+O(\rho^2)$$

$$=-\sum_{k=1}^{n}\beta_k\ln(x_k)+\rho\frac{-\sum_{k=1}^{n}\beta_k\ln(x_k)\sum_{m=1}^{n}\beta_m\ln(x_m)+\sum_{k=1}^{n}\beta_k\ln(x_k)^2}{2}+O(\rho^2)$$

$$=-\sum_{k=1}^{n}\beta_k\ln(x_k)+\frac{\rho}{2}\left[\sum_{k=1}^{n}\beta_k\ln(x_k)^2-\sum_{k=1}^{n}\beta_k\ln(x_k)\sum_{m=1}^{n}\beta_m\ln(x_m)\right]+O(\rho^2)$$

$$=-\sum_{k=1}^{n}\beta_k\ln(x_k)+\frac{\rho}{2}\left[\sum_{k=1}^{n}\beta_k\ln(x_k)^2-2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\ln(x_k)\ln(x_m)-\sum_{k=1}^{m}\beta_k^2\ln(x_k)^2\right]+O(\rho^2)$$

(39)

$$\Leftrightarrow\qquad f(\rho)=-\sum_{k=1}^{n}\beta_k\ln(x_k)+\frac{\rho}{2}\left[\sum_{k=1}^{n}(\beta_k-\beta_k^2)\ln(x_k)^2-2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\ln(x_k)\ln(x_m)\right]+O(\rho^2)$$

This may be rewritten to the more elegant form

$$f(\rho) = -\sum_{k=1}^{n}\beta_k \ln(x_k) + \frac{\rho}{2}\left[\sum_{k=1}^{n}\beta_k(\sum_{m\neq k}\beta_m)\ln(x_k)^2 - 2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m \ln(x_k)\ln(x_m)\right] + O(\rho^2)$$

$$= -\sum_{k=1}^{n}\beta_k \ln(x_k) + \frac{\rho}{2}\left[\sum_{k=1}^{n}\sum_{m\neq k}\beta_k\beta_m \ln(x_k)^2 - 2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m \ln(x_k)\ln(x_m)\right] + O(\rho^2)$$

$$= -\sum_{k=1}^{n}\beta_k \ln(x_k) + \frac{\rho}{2}\left[\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\left(\ln(x_k)^2 + \ln(x_m)^2\right) - 2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m \ln(x_k)\ln(x_m)\right] + O(\rho^2)$$

$$= -\sum_{k=1}^{n}\beta_k \ln(x_k) + \frac{\rho}{2}\left[\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\left(\ln(x_k)^2 + \ln(x_m)^2 - 2\ln(x_k)\ln(x_m)\right)\right] + O(\rho^2)$$

(40)

$$\Leftrightarrow \qquad f(\rho) = -\sum_{k=1}^{n}\beta_k \ln(x_k) + \frac{\rho}{2}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\left(\ln(x_k) - \ln(x_m)\right)^2 + O(\rho^2)$$

Using equation (32), (33) and (40) it is seen that

(41)

$$\frac{1}{\nu}\left(\ln(y) - \ln(\gamma)\right) = -f(\rho)$$

$$\Leftrightarrow \quad \ln(y) = \ln(\gamma) - \nu \cdot f(\rho)$$

$$\Rightarrow \quad \ln(y) = \ln(\gamma) + \nu\sum_{k=1}^{n}\beta_k \ln(x_k) - \frac{\nu\cdot\rho}{2}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\left(\ln(x_k) - \ln(x_m)\right)^2 + O(\rho^2)$$

or more generally, employing equation (39)

(42)

$$\ln(y) = \ln(\gamma) + \nu\sum_{k=1}^{n}\beta_k \ln(x_k) - \frac{\nu\cdot\rho}{2}\left[\sum_{k=1}^{n}(\beta_k - \beta_k^2)\ln(x_k)^2 - 2\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m \ln(x_k)\ln(x_m)\right] + O(\rho^2)$$

$$= \ln(\gamma) + \sum_{k=1}^{n}\nu\cdot\beta_k \ln(x_k) - \sum_{k=1}^{n}\frac{\nu\cdot\rho}{2}(\beta_k - \beta_k^2)\ln(x_k)^2 + \sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\rho\cdot\nu\cdot\beta_k\beta_m \ln(x_k)\ln(x_m) + O(\rho^2)$$

Comparison of equation (42) with the general translog function:

(43) $$\ln(y) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln(x_i) + \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} \ln(x_i) \ln(x_j)$$

shows that this will be equal to equal to equation (42) if **(i)**

(44) $$\alpha_0 = \ln(\gamma)$$

(ii) if

(45) $$\alpha_i = v \cdot \beta_i$$

and thus

(46) $$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} v \cdot \beta_i = v$$

employing that the sum of the β values is equal to unity.

The third condition is **(iii)**

$$\alpha_{ij} = \rho \cdot v \cdot \beta_i \beta_j = \rho \cdot v \cdot \frac{\alpha_i}{v} \frac{\alpha_j}{v} = \rho \cdot \frac{\alpha_i \cdot \alpha_j}{v} = \rho \frac{\alpha_i \cdot \alpha_j}{\sum_{i=1}^{n} \alpha_i} \quad ; i \neq j$$

Thus

(47) $$\frac{\alpha_{ij}}{\alpha_i \cdot \alpha_j} \sum_{i=1}^{n} \alpha_i = \rho$$

Finally the fourth conditions is **(iv)**

$$\alpha_{ii} = -\frac{v \cdot \rho}{2}(\beta_i - \beta_i^2) = -\frac{v \cdot \rho}{2}\left(\frac{\alpha_i}{v} - \frac{\alpha_i^2}{v^2}\right) = -\frac{\rho}{2}\left(\alpha_i - \frac{\alpha_i^2}{v}\right) = \frac{\rho}{2}\left(\frac{\alpha_i^2}{\sum_{i=1}^{n}\alpha_i} - \alpha_i\right)$$

Rearrangement of this gives:

$$(48)\quad \frac{2\alpha_{ii}}{\frac{\alpha_i^2}{\sum_{i=1}^{n}\alpha_i} - \alpha_i} = \rho$$

The above derivation shows that the CES function given in equation (14) may for $\rho \in O(0)$ be approximated by the general translog function (43) given the restrictions (44)-(48) on the translog parameters. Thus to test whether the CES function is a reasonable fit to an observed dataset, this may *in certain cases*, i.e. if it is suspected that ρ is close to zero (or correspondingly if σ is close to unity) be done by fitting a translog function to data and test whether the conditions (47)-(48) are fulfilled for the translog parameters. *If* the translog is a good fit to data and *if* (47)-(48) are fulfilled it may be concluded that the CES function is a good fit to data, and the CES parameters may then be estimated by equation (44)-(48).

It is clear that to test the restrictions (47)-(48) may be a lengthy and laborious procedure. The testing procedure may be eased quit a bit by working with the form of the Taylor approximation to ln(*y*) given in equation (41). This form shows that the approximation may be written as:

$$(49)\quad \ln(y) = \ln(\gamma) + v\sum_{k=1}^{n}\beta_k \ln(x_k) - \frac{v \cdot \rho}{2}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\beta_k\beta_m\left(\ln(\frac{x_k}{x_m})\right)^2$$

Comparison of this form with the more general:

$$(50)\quad \ln(y) = \alpha_0 + \sum_{k=1}^{n}\alpha_k \ln(x_k) + \sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\alpha_{km}\left(\ln(\frac{x_k}{x_m})\right)^2$$

shows that these are equal if **(i)** the conditions given in (45) and (46) are fulfilled together with **(ii)**:

$$\alpha_{km} = -\frac{\nu \cdot \rho}{2} \beta_k \beta_m \quad ; \quad \forall k,m \quad \Rightarrow$$

$$\alpha_{km} = -\frac{\nu \cdot \rho}{2} \frac{\alpha_k}{\nu} \frac{\alpha_m}{\nu} \quad ; \quad \forall k,m \quad \Rightarrow$$

$$\alpha_{km} = -\frac{\rho}{2\nu} \alpha_k \alpha_m \quad ; \quad \forall k,m \quad \Rightarrow$$

(51) $$\frac{\alpha_{km}}{\alpha_k \alpha_m} = -\frac{\rho}{2\nu} \quad ; \quad \forall k,m$$

This last identity (51) shows that a necessary conditions for (50) to represent the Taylor approximation to the CES function is that the factors $\alpha_{km} / (\alpha_k \alpha_m)$ are equal to the same constant for all *k,m*, i.e. especially that the factors are equal for all *k,m*. This is quit an easier condition to test than the conditions (47) and (48) for the direct translog approximation.

Thus an alternative to testing whether a given data set complies with a CES function is firstly to calculate the identities $\ln(x_k/x_m)^2$ (or alternatively $(\ln(x_k)$- $\ln(x_m))^2)$ for all *k,m*, together with the identities $\ln(y)$ and $\ln(x_k)$ for all *k*. Of course this gives a bit more introductionary labour, but compared with what is saved when testing the restriction (51) compared with testing the restrictions (47) and (48) the second method is believed to be the easiest to use. If the identity (51) is fulfilled in the second method, and if the fit of the function (50) to data is generally good it may the be concluded that the CES function is a good fit to data, and the translog parameters may then be estimated from (45), (46) and (51).

Corollary 1
The CES function given in equation (14) is equal to the Cobb-Douglas function

(52) $y = \gamma \cdot \prod_{i=1}^{n} x_k^{\nu \cdot \beta_k}$

in the limit ρ=*0* (corresponding to σ=*1*).

This is shown by combining equation (32), (33) and (34), as

$$\ln(y)\,|_{\rho=0} = \ln(\gamma) - \nu \cdot \lim_{\rho \to 0} f(\rho)$$
$$= \ln(\gamma) + \nu \cdot \sum_{k=1}^{n} \beta_k \cdot \ln(x_k)$$

Equitation (52) is then obtained by taken the exponential of both sides of this identity.

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