A model of mathematics teacher knowledge and a comparative study in Denmark, France and Japan

Winsløw, Carl; Durand Guerrier, Viviane; Yoshida, Hiroaki

Published in:
Annales de didactiques et de sciences cognitives

Publication date:
2010

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Winsløw, C., Durand Guerrier, V., & Yoshida, H. (2010). A model of mathematics teacher knowledge and a comparative study in Denmark, France and Japan. Annales de didactiques et de sciences cognitives, 15, 141-166.

VIVIANE DURAND-GUERRIER, CARL WINSLØW & HIROAKI YOSHIDA

A MODEL OF MATHEMATICS TEACHER KNOWLEDGE AND A COMPARATIVE STUDY IN DENMARK, FRANCE AND JAPAN

Abstract. A model for mathematics teacher knowledge based on the anthropological theory of didactics is presented together with a methodological discussion of how to *assess* such knowledge in practice. To this end we propose a concrete method involving "hypothetical teacher tasks" and individual as well as collaborative work of the teachers to be assessed. This discussion is illustrated by a small scale comparative study of how future lower secondary mathematics teachers (just about to graduate) from Denmark, France and Japan approach two hypothetical teacher tasks (related to teaching geometry and arithmetics).

Résumé. Un modèle des connaissances de l'enseignant des mathématiques et une étude comparative du Danemark, de la France et du Japon. Nous présentons un modèle pour décrire les connaissances des enseignants des mathématiques, avec une discussion méthodologique de son application pour évaluer les connaissances en pratique. Nous proposons une méthode concrète impliquant un travail individuel et collectif des enseignants évalués sur des « tâches hypothétiques d'enseignant ». Cette discussion est illustrée par une étude à petite échelle, où nous comparons le travail de futurs enseignants de collège (à la fin leur formation) sur deux tâches hypothétiques d'enseignant relevant de l'enseignement de la géométrie et de l'arithmétique).

Mots-clés. *Connaissances professorales, comparaison internationale, similarité, Thalès, formation des enseignants, Danemark, France, Japon, théorie anthropologique du didactique*

> And, upon the whole, a proof of a person's having knowledge is (…) the ability to teach; and for this reason we consider art, rather than experience, to be a science; for the artist can, wehereas the handicraftsmen cannot, convey instruction. (Aristotle, 1991, 13).

1. Introduction

What does a mathematics teacher need to know, and how should preservice education prepare future teachers? These two questions are closely related, as preservice education remains the main form of teacher education in most countries in the world. Indeed, the two questions are increasingly raised in an international perspective, as a consequence of the increasing interest in international comparison of school mathematics performance. And in fact the two questions are within the core of a *domain where many of the*

ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES, volume 15, p. xx - xx.

unresolved issues that were prevalent 25 years ago remain unresolved (Alter and Pradl, 2006, 42). A recent American report on teacher education research asserts the "relative thinness" of existing research, and suggests in particular that *research on teacher preparation defines more precisely the questions that need to be addressed and the data that need to be gathered* (ECS, 2003, 7).

We believe that a central part of this "black hole" (Altar and Pradl, 2006) of teacher education research is a lack of dependable models and methods to describe and assess teacher knowledge. This study is made to contribute, for the case of mathematics teacher education, to fill this gap. We deliberately use the term "knowledge", rather than e.g. competencies or skills, because we focus here on possible contributions of pre-service education – not on what could be gained by experience, in service training and so on. This does not mean that our interest is limited to knowledge in the academic, official sense. Even pre-service education may develop experience-based knowledge of teaching through various forms of practice integrated in the educational program, and of course students have beliefs about teaching which come from their own experience as pupils. But in this study we do not envisage the full complexity of the link between initial education and teaching. Our primary focus is on explicit knowledge that newly formed teachers can mobilise in front of a "hypothetical" teaching situation. In fact, we did this study in three rather different settings (Denmark, France, Japan) in an attempt to eliminate the idiosyncracies of local educational systems and cultures. Local conditions can, of course, not be ignored in general and in practice; but in this study we deliberately attempt to go beyond them.

The paper presents three distinct but interrelated parts:

- a theoretical model for mathematics teacher knowledge based on the anthropological theory of didactics initiated by Y. Chevallard,
- a methodology for assessing mathematics teacher knowledge based on what we call *hypothetical teachers tasks*, exemplified by two tasks related to two particular domains of mathematics teacher knowledge
- results from a comparative study using these tasks, involving 30 graduating teacher students in Denmark, France and Japan (the countries of the authors, representing quite different systems of training lower secondary teachers).

Regarding the last point, the data for Denmark and France were briefly discussed by Winsløw and Durand-Guerrier (2007), in connection to a broader comparison of their teacher education systems. In this paper we provide a detailed analysis of the data for three countries based on a more precise model for teacher knowledge, that we now proceed to explain.

2. Mathematics teacher knowledge: the anthropological approach

It is widely acknowledged that teachers need to know the contents they are teaching, and that they need to know more than this. The folklore wisdom is that teachers need to know some more of the related contents – usually, more of the relevant scholarly disciplines – than the students (although *how* much is often debated). It is also usually admitted that one may and should know something about teaching, at least from experience; this kind of knowledge is sometimes called *pedagogy*. Indeed, a number of teacher education programmes have the relevant academic discipline(s) as the main course, and "pedagogy" (sometimes labelled education, educational psychology or the like) for dessert.

This state of affairs subsists despite rather well established tendencies in research that seem to suggest a different approach. In a much quoted paper, Shulmann (1986) advocates that in addition to content knowledge in the regular form, the teacher needs two supplementary forms of it: *pedagogical content knowledge* about how to teach the contents, and *curricular knowledge* concerning the educational programmes and materials for teaching a given set of contents. The most important insight here is not, perhaps, the categorisation, but the stipulation that the essential of teacher knowledge is a *teaching-oriented extension, or deepening, of plain content knowledge*. In the last 20 years, this idea has been quite influential in the anglophone research literature on mathematics education and in particular on mathematics teacher knowledge (for two prominent examples, see Ball, 1991, Ma, 1999). Another important emerging idea is that general theories and concepts related to teaching methods become more useful, and take on (new) meaning, when they are used in the context of specific subject matter contexts (see e.g. Ball and Bass, 2000).

These ideas seem close to a relatively well-established tradition in continental-European didactics, namely the *didactical study of particular contents* (what the Germans call *Stoffdidaktik*). Here, the structure and uses of school mathematics are studied in great detail in view of improving the corresponding teaching (textbooks, classroom activities, problems etc.). Besides such an "a priori analysis" of the mathematical contents, studies in this tradition are often supported and driven by extensive experimental interventions (cf. e.g. Brousseau, 1997). It is interesting to note that a similar emphasis on didactically driven study of the contents to be taught, as a crucial step in so-called *lesson studies*, is found in Japan (cf. eg. Shimizu, 1999, 112).

However, in order to assess mathematics teacher knowledge in a systematic and controlled way – even if this will necessarily be in partial and "local" ways – we need a more precise model of *what is assessed*, i.e. an operational *epistemological* model for what mathematics teachers need to be able to do. As the above discussion suggests, this will include – or even start with – an activity oriented model of *mathematical* knowledge, and it will also have to include other aspects of the conditions for the teachers' work. To make it useful in a comparative setting, we must also be able to take institutional and cultural constraints into account, if not for other reasons, then in order to assess parts of teacher knowledge which can said to be relatively independent of such constraints.

To answer this need, we are convinced that the recent developments in anthropological theory of didactics (hereafter abbreviated ATD) furnish a promising basis. While the ATD literature (e.g. Chevallard, 1999; Bosch and Gascòn, 2002) should of course be consulted for a fuller account, we now explain the notions from ATD as we use them in this paper.

The central idea of ATD is to model human activity as responses to *types of tasks*, such as found in daily life (e.g. cook an egg) or scholarly subjects (e.g. find the product of two given integers). An important companion to a type of task is a *technique* or method to carry out the task. The technique is often enforced by a tacit routinisation of a frequently encountered task, but it could also be an object of explicit instruction. More generally, it is an important characteristic of human activity to allow for coherent discourse about tasks and techniques (called *technology*), and in some cases to organise these discourses in *theories* that make explicit the understandings and justifications underlying technology and techniques. For instance, an instruction on how to perform a multiplication integers belongs to a technology, while the systematic discussion of why the multiplication methods works is within the domain of theoretical discourse. A *punctual mathematical organisation* or *praxeology* consists of these four elements: a type of task, a technique, a technology and a theory, where each element corresponds to the previous one.

Notice that any *description* of a praxeology will by definition be situated at the technological or theoretical level; indeed, task types and techniques belong as phenonema to the level of practice and they are often based on tacit knowledge. A technique for multiplication that involves “pencil and paper” may look more explicit than one which is based on the operation of an abacus or a handheld calculator, but *a priori* they are both observable activities which *can* be described and justified in different ways, but need not be. Thus the *practice block* (task type, technique) may exist independently of the *techno-theoretical block* (technology, theory). Also, in many instances, a person enacting a praxeology, such as cooking eggs or multiplying integers, may enact some technology, such as simple instructions, while having no wider theory to explain the practice.

Human practices are interrelated and *organised*. Isolated practices, called *punctual organisations*, contain just one type of task, but they very often team up in *local organisations* which are characterised by employing a common technology (such as a system of language and symbolism related to a set of practices in arithmetics). In the presence of a theory, local organisations may be further unified in *regional organisations* which are collections of praxeologies sharing a common theory (e.g., a theory of arithmetics). In fact, *mathematical organisations* (abbreviated MO) are often highly structured and stratified in principle, while it is still possible and common for users to enact them only at the punctual or local level. To the analyst, a *reference model* – i.e. a description of a regional mathematical organisation – may then be useful to describe and analyse the practices observed. A good example of this, in the context of mathematical organisations enacted in Spanish highschool, is given by Barbé et al. (2005).

While we have many good examples of how to analyse mathematical organisations using the above theory, its use in the context of *teaching practices* remains less developed. Bosch and Gascòn (2002) suggests that teaching practices should be considered as *didactical organisations* (abbreviated DO), where the task types refer to tasks of the teacher. Such a task type could be, for instance, *plan a lesson on multiplication of two-digit integers.* As the example suggests, a DO may be closely and explicitly related to a MO, and such a DO can be viewed *essentially as an answer to the question “How does one establish a MO* [for students]*”* (Bosch and Gascòn, 2002, 35). In general, mathematics teacher knowledge is then enacted in DOs, with the knowledge being *articulated* in their techno-theoretical blocks.

As a matter of fact, the literature does not provide us with extensive examples of descriptions of DOs, and even less examples of their use in analysing or developing teaching practice. Moreover, if the concept of DO is to include all aspects of the mathematics teachers' practice, not all punctual DOs can be related directly to a MO. Evidently, organising and managing a classroom imply, at least to some extent, techniques and perhaps also technology and theories which are transversal to the MOs enacted. And even if we restrict ourselves to didactical task types directly related to a MO, a local DO could be structured in quite different ways, above all with respect to the place of the tasks in a sequence of teaching activities, such as lesson planning, teachers' tasks related to different phases of classroom teaching (sometimes called *didactical moments*), homework grading, and so on. Notice that these categories are in themselves particular to DOs and "transversal" to MOs.

With this, we are thus back to the discussion at the beginning of the section, in the following sense: how do MOs and DOs interact? Here is our model, in short: a local DO consists of a family of punctual DOs, which in a teaching activity will be enacted consecutively in time. We can think, for instance, of a local DO as a model of the teachers' activity in relation to a sequence of lessons which he considers as a "whole"; the common technology relates to the aims of such a teaching unit. Some of the task types (defining the punctual DOs) relate directly to a MO, for instance a DO task type may be to construct a question for students that will enable them to work on the MO. The teacher employs, to solve the task of a given punctual DO, a technique which is at least potentially *explained* by the overarching technology; the latter will then also refer to the MO in case the task type is related to it.

The students' work on a particular MO may in practice co-exist with several other activities which they are supposed to enact (including organisations transposed from other scientific disciplines, but also "behavioral organisations" such as manners of interaction among students and teachers). The idealisation – and reduction of complexity – which we want to make in this study, is to assess mathematics teacher knowledge while minimising the impact of other aims and constraints of teaching than those pertaining to the learners enaction of certain MOs. That is, we want to describe and assess primarily those parts of DOs which are related directly to certain MOs. This suggests that the “concrete setting of teaching” for the DO has to be replaced with a simplified, “hypothetical” local DO.

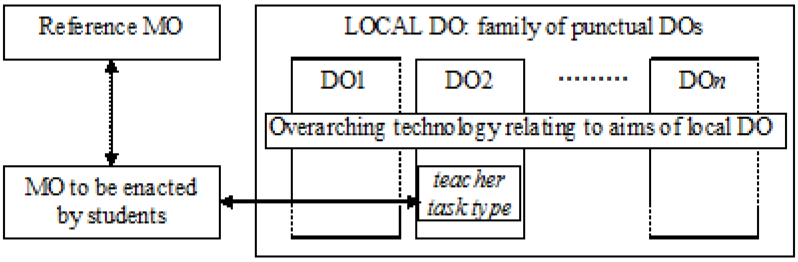

Fig.1. *A teaching sequence of punctual DOs, some of which relate directly to a MO. The analysis of the punctual DO employs a reference model of the local MO.*

In a sense, one can *observe* DOs directly, through their plain enaction, in teaching activity. We have already explained why this is not suitable in this study, where we want a clear picture of teacher students’ reactions to identical and simplified teacher tasks. It is important to note that what we can then record is just the respondents’ *technology* and perhaps *theory* as probed by a *description* of these tasks. On the other hand, the interaction among *teachers* is an important real-life channel for the development and exchange of didactical technology (and to some extent, theory). This kind of interaction can be enabled by arranging teacher students’ discussion of the tasks, rather than getting their “answers” in writing or in individual interviews. And from this complex discourse, we can then try to extract the key techniques which they are likely to use in an interaction with students or other forms of actual teaching activity.

We pause here to emphasise the current impossibility to build reference models for empirical research on DOs which are precise regarding the techno-theoretical block. This is true even for work within a single country, and *a forteriori* in a comparative study. By contrast, constructing operational reference models may be an almost trivial task for a MO (cf. above). This illustrates the current lack, for the teaching profession, of a common *professional language* and of widely known and acknowledged theoretical models; in short, for shared, dependable teacher knowledge.

To assess mathematics teacher knowledge outside of the classroom, we will then present pairs of respondents with certain tasks coming from a punctual DO that is situated in a hypothetical context, which is however susceptible of being recognised as meaningful by the respondents (and, in a wider sense, actually be so). The analysis of responses to such tasks will necessitate the construction of reference models of both MO and DO (the latter being mainly focused on didactical techniques). We present two examples of this procedure in the next section, and results from use of them in the following section.

3. Hypothetical teacher tasks: two cases

The two tasks used in this study are the same as those published in (Winsløw & Durand-Guerrier, 2005, appendix; for the sake of completeness they are reprinted in this section). We now present a thorough analysis of them, using the theoretical model presented in the previous section. Notice that they are both in the context of lower secondary school, and relate to different major regional MOs which are taught in most countries at this level (geometry, algebra).

3.1. Teaching similarity or proportions (HTT1)

The hypothetical teacher task (HTT1, cf. below) centers around a *mathematical task* of the following type:

> T_1: given a triangle Δ with sides a, b, c known, and a number $a'>0$, find b' and c' such that the triangle with side lengths a', b' and c' is similar to Δ.

> In fact, the student task contained in HTT1 is a variant of T_1, where the mathematical notions *triangle* and *similar* are concealed in a “real world” setting: what is given is the distances (3,3,4) between three points on an aereal photo, and the corresponding points on a magnification – which is implicitly a similar figure – to be constructed, in which the longest distance (corresponding to the side which is 4 on the photo) is also given. While the recognition that three distinct points corresponds to a triangle is probably rather straightforward, some students may not recognise the theoretical concept of similar triangles in this task, but rather a special case of T_1:

T_{11}: given a triangle Δ with sides *a, b, c* known, and a number $a'>a$, find b' and c' such that the triangle with side lengths a', b' and c' is a "magnification" of Δ.

HTT 1 (translated from the Danish/French/Japanese versions used in the study)

You assign the following task to your 8th grade pupils:

An aerial photo is used to draw a map. To begin with, three points are marked on the photo; the distances between these points are 4 cm, 3 cm, and 3 cm. The map must be slightly larger than the photo: the longest distance between the three points should be 6 cm on the map. What should the other two distances be on the map?

Some pupils answer: "5 cm and 5 cm"; others say: "4.5 cm and 4.5 cm".

First task for the teacher (*to be solved individually within 10 minutes*) Analyse the solutions. What would you do as teacher in this situation? Please take notes.

Second task for the teacher (*to be solved in conversation with the other teacher student, 20 min.s*) Please, discuss your ideas with respect to using this situation to further the pupils' learning.

In both cases the technique to solve this task is

τ_1 : first compute $k = a'/a$, then b' can be found as kb, and c' as kc.

Notice that τ_1 is just a implicit form of "action", which – when described, as above – becomes already part of a technology. The technology could also be an explanation of the kind

θ : in similar triangles, the ratios of corresponding sides are equal: $a'/a = b'/b = c'/c$.

This technology may relate a number of other practice blocks, corresponding to task types such as

T_{12}: given two different triangles, determine if they are similar.

T_{13}: as T_1, but with a quadrilateral (τ_3 involving subdivision into two triangles).

Finally, the *theory* involved in justifying θ and developing its relations with other technologies, could be the

Θ : theory of proportions in Euclidean geometry (including Thales' theorem)

but this is by no means uniquely determined by the previous elements; less "formal" arguments about similar figures might also form a theoretical background for θ. And, particular at more advanced levels, one could find technologies akin to θ within a geometrical theory of linear maps M: $\mathbf{R}^n \to \mathbf{R}^n$ (in fact $n = 1$ sufficces, but $n = 2$ is also relevant). Indeed, the formulation of the task for the pupils leaves this quite open, but in working with HTT1, respondents will situate it within some kind of technology that also implicitly suggests a background theory of some kind.

With such a choice of θ and Θ, the quadruple (T_1, τ_1, θ, Θ) forms a punctual MO corresponding to T_1, and a local MO is then a family (T_i, τ_i, θ, Θ) where θ enables the articulation of all the practice blocks (T_i, τ_i). It is not given beforehand if the hypothetical DO aims just at the punctual MO determined by T_1, or if it may have a larger scope with the local MO determined by θ, or even the regional MO determined by Θ.

The didactical type of tasks, to which HTT1 belongs, leaves this open. We may describe this task type as follows:

T_1*: given different student responses to a task of type T_1, determine what to do as a teacher to make students learn.

The openness of HTT1 can be interpreted as the object of the verb "learn": are we talking about a technique like τ_1 (punctual MO, where the tasks could be varied within T_1), about the technology θ (local MO, where the variation is over practice blocks), or even the regional MO where different technologies within the scope of a theory Θ could be involved? However, T_1* has another kind of constraint, which is characteristic of "critical didactical decisions" in the classroom: that of *didactical time* (Chevallard, 1991). Respondents are likely to think of what can be done for the students learning within a lesson, in a broad sense; thus an uncontrolled variation within the scope of a theory Θ may not lead to a pertinent *didactical* technique to activate for T_1*. Indeed, it seems necessary to address the fact that some students did not provide a correct answer to the given task of type T_1; in particular, to identify a technique that could led to the false answer, such as

τ_1^- : first compute $k = a' - a$, then b' can be found as $b + k$, and c' as $c + k$,

which, by explicitation and generalisation, corresponds to the mathematically incorrect technology

θ^-: in similar triangles, the differences of corresponding sides are equal: $a' - a = b' - b = c' - c$.

A possible aim of variying the tasks could then be to confirm if some students apply τ_1^- , to have them formulate something like θ^-, and finally convince them (through appropriate tasks) that θ^- is errouneous in the sense that triangles constructed using τ_1^- are in fact not similar. One might also think of ways to demonstrate that θ^- is not compatible with Θ.

This leads us to some of the *didactical* techniques that could be applied to solve tasks of the type T_1*. To describe and motive these techniques, we first formulate some concrete elements of teachers' reflection about the students' answers to the task (of type T_1); they relate in rather directly to the teachers' command of (T_1, τ_1, θ, Θ):

S_1: Identify correct student answer, 4.5 cm, e.g. by using τ_1 (no explicit technology).

S_2: Identify correct student answer, 4.5 cm, stating that it is consistent with θ (possibly without explicit link to theory justifying θ).

S_3: Identify correct student answer, 4.5 cm, stating that it is consistent with θ, and refer to appropriate theory (e.g. "Thales theorem") to justify the principle of θ.

S_4: Identify wrong student answer, 5 cm, e.g. using τ_1 (no explicit technology needed).

S_5: Identify wrong student answer, 5 cm, stating that it is consistent with a "technology" like θ^{-} (possibly without giving an example or theoretical reason to reject θ^{-}).

S_6: Identify wrong student answer, 5 cm, stating that it is consistent with θ^{-}, and give an example or theoretical reason to reject θ^{-}.

These elements of solution come out of – and help to recognise – the following elements of didactical techniques (some of which can clearly be generalised beyond T_1*):

τ_{10}*: find answer to pupils' task (here, recognised in solution elements S_1 and S_4)

τ_{11}*: identify appropriate technology for the students' task (here, S_2)

τ_{12}*: identify the techniques and technologies that could underlie the wrong answers (here, S_5)

τ_{13}*: identify reasons to reject the technology leading to the wrong answers (here, S_6)

τ_{14}*: identify reasons to accept the appropriate technology (here, S_3)

Notice that τ_{11}* and τ_{12}* both presuppose τ_{10}* in the sense that identifying technology requires knowing a technique. Likewise, τ_{13}* presuppose τ_{12}*, and τ_{14}* presuppose τ_{11}*.

To proceed in class, some of the following techniques may then be proposed. They are based on the teachers' broader grasp of the MO, and aim to develop the students' understanding of it at different levels:

τ_{15}*: present several explanations (technologies, examples, reasonings) to the pupils;

τ_{16}*: organise a class discussion about the two solutions, to have students realise what is the correct answer

τ_{17}*: have pupils work on more instances of T_1

τ_{18}*: actively and explicitly base activities on students' knowledge and previous experience with this or related local MOs

τ_{19}*: actively and explicitly make use of different forms of representation (diagrams, tables, formulae etc.) in activities or presentation related to θ.

τ_{19a}*: organise some activity with technology (geometry software, to work on similarity – with the given or with other examples).

These "techniques for action" depend heavily on τ_{10}*,…, τ_{14}*. For instance, τ_{17}* could mean just to provide a number of exercises allowing pupils to train τ_1; but in the presence of τ_{13}* it could mean carefully selected examples that would allow the pupils to realise that θ^- is false.

A *technique* τ_1* to solve a task of type T_1* such as HTT1 could then be described a subset of {τ_{10}*,.., τ_{19}*, τ_{19a}*}; it may of course involve other partial techniques, but the above are the elements we have found most crucial and which we use in our actual coding of solutions. A *didactical technology* θ* for describing such techniques has just been presented, even within the setting of a theoretical framework (ATD, cf. section 2). As we mentioned in Sec. 2, the technology found in teachers' discussions about T_1 is likely to be different and to exhibit considerable variations, but it will still contain elements that can be meaningfully identified with the techniques above. For instance, a major technological component in explaining the above techniques is related to *how the pupils' understandings is addressed* – does the technology enable to go beyond the recognition of right and wrong answers (and even methods)? How precisely can different approaches to T_1* be described and assessed? and, of course, for the latter question, on what theoretical basis? Just to cite one possible source of techno-theoretical component corresponding to (T_1*, τ_1*), we mention Brousseau's discussion of the "puzzle situation" (Brousseau, 1997, 177ff).

HTT 2 (translated from the Danish/French/Japanese versions used in the study)
Your pupils are working in class (grade 9). They encounter at some point the need to calculate the expression $(-2)(x-3)$ to produce the expression $-2x+6$. A pupil, whom you know to be rather gifted, calls on you and says: “look, I have arrived at this” [he points the expression

$$-2x+(-2)\cdot(-3)$$

in his notes]. “I know the last term should be 6. But then I began to doubt – why is it so?”

The lesson is about to end. You decide to postpone the question until the lesson tomorrow, and say: “Yes, that’s a good question. Let us come back to that tomorrow”.

Teacher task (*to be discussed in pairs of teacher students, in 20 minutes*)**.** Imagine you are in the teachers’ lounge, and discuss the problem with your colleague: how could you make this question an opportunity to learn?

3.2. Reviewing negative numbers: a tedious “gap” (HTT2)

The question posed by the gifted pupil in HTT2 is one that could in principle come up as negative numbers and their arithmetics is introduced, in many countries several years before the hypothetical context (grade 9). The pupil perhaps asked it then, but did not understand or accept the explanation he got. This was the case of the French author Stendhal:

> Imagine how I felt when I realized that no one could explain to me why minus times minus yields plus… Mr. Chabert, whom I pressed hard, was embarrassed. He repeated the very lesson that I objected to and I read in his face what he thought: “It is but a ritual, everybody swallows this explanation. Euler and Lagrange, who certainly knew as much as you do, let it stand. We know you are a smart fellow… It is clear that you want to play the role of an awkward person… It took me a long time to conclude that my objections to the theorem: *minus times minus is plus* simply did not enter M. Chabert’s head, that M. Dupuy will always answer with a superior smile, and that mathematical luminaries that I approached with my question would always poke fun at me. (quoted in this translation by Hefendehl-Hebeker, 1991, p. 27).

The pupil of HTT2 is now in ninth grade; would he get a satisfactory answer to his question? If he does not, will he give up his interest in mathematics, like the hero of Stendhals autobiographic memories cited above? We don't know; the 9th grader is fictive. But certainly the question arises in similar ways to many pupils, and continues to haunt even some adults, like Stendhal. What should the teacher know to go beyond repeating his lesson?

As before, let us first analyse the MO to which the teacher task relate. At the basis we have the task type

> T_2: Determine the product of two negative numbers which comes with the well known technique
>
> τ_2: just compute the product of the corresponding positive numbers, which in fact may be carried out rather automatically.

An official technology is

> θ : For any real numbers a and b one has $(-a)\cdot(-b) = ab$.

A theoretical environment Θ for this statement could be any axiomatic theory of the arithmetic of the reals, the main ingredient in a formal proof being the distributive and commutative laws (see e.g. τ_{25}* below). In particular, this is likely to be the essence of the "lesson" which Stendhal (cf. quote above) would not accept. Indeed, the *if-then* nature of a theorem in an axiomatic theory – where the truth of all statements is relative to some a priori "arbitrary" axioms – is not easily digested in grade 9. This is not only because mathematics is usually not presented that way at the level of 9th grade, but also because the "lesson" it likely to treat Θ in a partially implicit way – perhaps presenting the axioms as "evident rules", or just use them implicitly. More crucially, there does not seem to exist convincing intuitive explanations for this rule, unlike what is the case for most common arithmetic principles like the distributive law (e.g. use areas of rectangles) or $-(-a) = a$ (e.g. use "sign corresponds to reflection in 0 on the number line"). And finally, there does not seem to be simple applications in real life of *products* of two negative numbers, unlike products which have a positive factor (could be though of as repeated addition etc.)

Considering the use of explanations and reasonings adapted to the pupil, we are therefore approaching the demanding *didactical* task type

T_2*: A pupil masters (T_2, τ_2) but asks you why that technique is correct. Determine how to make use of this question as an opportunity to learn.

As in T_1* it is open what is the object of the verb "learn", in particular what technology and possibly theory related to (T_2, τ_2) the teacher should make use of. This should be determined from the hypothetical context. In this task the subject of "learn" is also open: *who* could or should learn from this, just the pupil asking the question, or the whole class?

Despite the difficulties of exhibiting a convincing justification of θ at the given level, we can nevertheless think of the following didactical techniques, some of which make use of the fact that the pupil seems convinced that $(–2)\cdot(x–3) = –2x + (–2)\cdot(–3)$; but in other cases the same techniques could be used after checking that the pupils accept such uses of the distributive law. The first concern direct mathematical explanations of θ, aimed at the questioning pupil or the whole class:

τ_{21}*: explanation using "number patterns", e.g. look at $n\cdot(–3)$ for n = 2,1,0,…

τ_{22}*: explanation by drawing lines, e.g. $y = –2x$, when the pupil is convinced this is the equation of a real line (then drawing first the halfline for $x > 0$ forces $–2x$ to be positive for $x < 0$).

τ_{23}*: explanation based on "parenthesis magic", such as $(–2)\cdot(–3) = –(2\cdot(–3)) = –(–6) = 6$

τ_{24}*: present a more or less complete proof based on the distributive law, e.g. as follows:

$0 = 2\cdot(3–3) = 2\cdot3 + 2\cdot(–3) = 6 + 2\cdot(–3)$; so $2\cdot(–3) = –6$;

and as $0 = (2–2)\cdot(–3) = 2\cdot(–3) + (–2)\cdot(–3)$, we conclude that $6 = (–2)\cdot(–3)$.

τ_{24a}*: explanation based on the equation $(–2)\cdot(x–3) = –2x + (–2)\cdot(–3)$, e.g. as follows:

when $x = 3$ the left hand side is 0 and so $6 = 2x = (–2)\cdot(–3)$.

τ_{25}*: circular or otherwise inconvincing explanations, e.g. inappropriate real life explanations (such as "twice you remove a deficit of 3€

from my account"), travelling forth and back on the number line with negative speed, etc.

τ_{25a}*: explanations appealing explicitly to external authority – of the teacher ("trust me"), of convention ("this rule is used by everyone"), of technology ("try out what your calculator says") etc.

τ_{25b}*: after proposing τ_{25}* and/or τ_{25a}*, realising the insufficiency of the technique.

Notice that the last three techniques are not really related to the MO described before, and could be used as a way to cover a poor understanding of it. But of course, a teacher may also honestly think that the pupil can not learn anything on this question, except that mathematics contains conventions which one just has to accept.

To proceed in class, some of the following techniques may then be proposed. They are based on the teachers' broader understanding of the MO, and aim to develop the students' understanding of it at different levels:

τ_{26}*: present several explanations (technologies, examples, reasonings) to pupils;

τ_{27}*: organise a class discussion in order to have pupils find explanations or otherwise explore;

τ_{27a}*: have pupils work on more examples of T_2, for example based on τ_{21}*;

τ_{27b}*: organise some activity with technology (calculators, spreadsheet…) to explore the arithmetics of negative numbers, for example based on τ_{21}*;

τ_{28}*: actively and explicitly base activities on students' knowledge and previous experience with this or related local MOs

τ_{29}*: actively and explicitly make use of different forms of representation (diagrams, tables, formulae etc.) for example based on τ_{22}*.

In practice, the techniques mentioned above may be just partial in the sense that they do not in themselves enable a reasonable solution to T_2*. For instance, a teacher may express her intention to present several explanations (and even justify it e.g. with reference to different styles of learning), but be unable to do so in the concrete case.

Besides discussing the different techniques for responding to T_2*, some more specific features of HTT1 may be discussed by teacher students, e.g. whether the question should be taken up just with the inquiring pupil, or with the whole class.

3.3. A tentative quantitative measure based on the above.

A full presentation of the techniques developed by respondents (among those designated above) provides rather detailed information, but as some techniques are more appropriate than others – and some are decisively faulty – we may try to summarise the overall performance by designating points to techniques in the following manners:

> 2 points: the technique is entirely appropriate and could contribute to pupil learning;
>
> 1 point: the technique might be appropriate and could possibly contribute to pupil learning;
>
> 0 point: the technique is not approproate and could not further pupil learning (attributing 0 points for such a technique is of course debatable, and amounts to counting only appropriate techniques, without "punishing" faulty ones).

While the techniques proposed can be rather objectively identified in the students teachers' discussion, any assignment of points is of course somewhat normative.

In Fig.2-3 is our suggestion for such an assignment. Notice that we have assigned 0 point to τ_{25}* and τ_{25a}*, which are potentially causing obstacles to pupils learning, but as τ_{25b}* *always* comes with one of these and tend to neutralise this effect, the latter has been assigned 1 point (it may be that teachers who, in virtue of τ_{25b}*, realise the defaults of τ_{25}* and τ_{25a}*, may finally avoid them and use other techniques).

	τ_{10}*	τ_{11}*	τ_{12}*	τ_{13}*	τ_{14}*	τ_{15}*	τ_{16}*	τ_{17}*	τ_{18}*	τ_{19}*	τ_{19a}*
Points	1	2	2	2	2	1	1	1	2	1	1

Fig. 2a. *Tentative grading of techniques for T_1*.*

	τ_{21}^*	τ_{22}^*	τ_{23}^*	τ_{24}^*	τ_{24a}^*	τ_{25}^*	τ_{25a}^*	τ_{25b}^*	τ_{26}^*	τ_{27}^*	τ_{27a}^*	τ_{27b}^*	τ_{28}^*	τ_{29}^*
Points	1	1	1	2	2	0	0	1	1	1	1	1	2	1

Fig. 2b. *Tentative grading of techniques for T_2*.*

One objection which could be raised to such a grading is that teacher student pairs who develop a wide range of possibly mediocre techniques would get more points than ones who develop a single, appropriate plan for intervention in the situation. However, in the setting of the tasks, exploring a richer variety of (potentially appropriate) techniques would, indeed, seem an indicator for the quality of didactical interventions they might, eventually, be able to implement.

4. A small-scale comparative study

We now present the results of a comparative study in Denmark, France and Japan, based on HTT1 and HTT2 and the coding and assessment schemes explained in Sec. 3. The two tasks are about mathematics teaching at the lower secondary level – more precisely, grade 7 and 9 respectively – and it is affirmed that the two tasks corresponds to programs and common teaching pratice in the involved countries and at the indicated grade level. In this study, respondents were students who were very close to obtaining, or had just obtained, the degree that enables them to take up a position as teacher in grade 7 through 9 at schools in the respective countries.

We had 5 pairs of teacher students (a total of 10 respondents) do the two tasks in each country. With such a small sample, it did not make sense to aim for formal representativity across institutions in each country; and in fact, all 10 students in each country graduated, or were about to graduate, from the same institution. While they were all in larger cities, there is no reason to believe that these institutions are special or significantly different from other institutions responsible for teacher training in those countries – except on two points: (1) the Danish and Japanese institutions are "upper end" in the sense that they are sought by relatively well performing students; (2) while an equal number of male and female respondents participated in Denmark and France, only females participated in Japan. In all three countries, participation was voluntary, but the sample of volunteers within the institution was quite arbitrary and noone refused to participate.

Indeed, it is our impression that the 10 teacher students in each country can be considered, together, a reasonably "average" sample for the total target population within their institution. However, we readily admit that some of the variables mentioned should be controlled better in a large scale sample, and that the somewhat loose control of them here (in part enforced by pratical circumstances) does imply some reservation as to the significance of our results.

Tasks and oral introductions were translated from English or French into each of the two other languages involved (Danish and Japanese), and in general we made every effort to provide the same conditions for all 15 pairs: *verbatim* the same introduction, the same time slots allowed, paper and pencils available and a nice, quiet room, and of course, no further intervention on our part. The discussions were taperecorded, resumed and coded by one of the authors, and checked by at least one more author (only one author understands Danish, so here the checking for the Danish pairs was based on a translation). An example of coding for one pair is included in Appendix 1. The written materials produced by the teachers (during the first 10 minutes for HTT1, and possibly during discussions) were collected as supplementary evidence.

The coding corresponds to the didactical techniques presented in Sec. 3. We may then summarise the results for each country as the *number of pairs who, in their discussion, explicitly proposed each of these techniques (without clearly rejecting them in the sequel).* This is shown in Fig. 3a-b. These tables contain a significant amount of information about *what* techniques the teacher students develop during their discussions. Our stipulation that these techniques give a reasonable picture of the contents of these conversations can, of course, only be controlled by examining them in more detail; but we point out that while most of the techniques were identified *a priori*, we did carefully consider if something significant turned up unexpectedly, and in fact added one technique (namely τ_{24a}*) to our original list, based on the data.

HTT1	τ_{10}*	τ_{11}*	τ_{12}*	τ_{13}*	τ_{14}*	τ_{15}*	τ_{16}*	τ_{17}*	τ_{18}*	τ_{19}*	τ_{19a}*
DK	4	3	3	2	0	0	3	4	1	0	2
F	5	5	5	5	1	5	3	3	3	1	3
JP	5	5	5	3	0	4	3	4	1	0	2

Fig. 3a: *Number of teacher pairs proposing the didactical techniques while discussing HTT1.*

HTT 2	τ_{21}*	τ_{22}*	τ_{23}*	τ_{24}*	τ_{24a}*	τ_{25}*	τ_{25a}*	τ_{25b}*	τ_{26}*	τ_{27}*	τ_{27a}*	τ_{27b}*	τ_{28}*	τ_{29}*
DK	0	0	0	1	0	5	5	3	3	1	0	0	2	1
F	4	0	4	2	2	1	4	4	3	0	0	0	5	0
JP	0	1	4	1	1	4	1	5	5	1	2	0	2	0

Fig. 3b: *Number of teacher pairs proposing the didactical techniques while discussing HTT2.*

Let us point out just a few things which can be read out of the two tables:

(1) All pairs, except one Danish, identify correctly the right and wrong student answers in HTT1.

(2) All French pairs identify principles behind those answers, and want to provide several explanations of them to pupils. One pair even provide a theoretical reference (Thales' theorem). All Japanese pair explicitly state a principle behind the correct answer.

(3) The Danish pairs all propose little else than inappropriate explanations of $(-2)\cdot(-3) = 6$ (all propose both τ_{25}* and τ_{25a}*) and 3 recognise that they are inappropriate; also, most of the French and all of the Japanese pairs propose such explanations, but these pairs recognise their insufficiency and they also develop alternative techniques.

Activities based on information technology are not developed by any pairs for HTT2, while some pairs in all countries consider this for HTT 1 (τ_{19a}*).

In order to gain a more immediate overview of the data, we used the grading scheme from Sec. 3.3 to calculate for each country the total number of points obtained by the five pairs, on each task. The result is shown in Fig. 4a and visualised in Fig. 4b.

Points	HTT1	HTT2
DK	31	13
F	58	33
JP	46	26

Fig. 4a: *Overall score in the two tasks, for the 10 students of each country.*

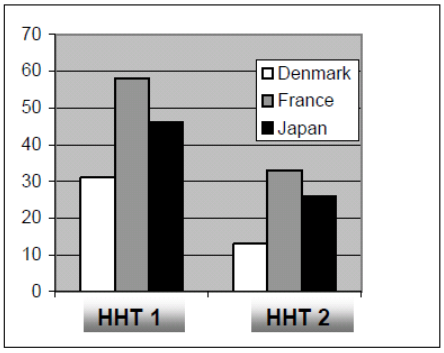

Fig. 4b. *Graphical representation of scores as indicated in Fig. 5.*

As is particularly clear in Fig. 4b, there is a common pattern for HTT1 and HTT2, even if the task types are rather different, not only in mathematical contents but also in terms of the hypothetical context. We want in particular to point out two general tendencies:

- with respect to developing more appropriate techniques, the French students do somewhat better than the Japanese students, and both groups are significantly ahead of the Danes;
- in all three countries, the overall "performance" is more satisfactory for HTT1 than for HTT2 (notice that although the sum of points for the two tasks can clearly not be compared directly, it seems evident that more appropriate techniques are proposed for HTT1).

While above-mentioned the reservations caused by methodology makes it inappropriate to consider the absolute scores as certain and representative measures for teacher students' performance on the two tasks, we affirm that these two *tendencies* reflect very well our overall assessment of our data for the three countries.

5. Discussion of the results

Teacher education is undergoing frequent reforms in many countries, often with little evidence to support change. This is not only true at the global, institutional level, but also more locally in the choice and organisation of courses in the teacher education programme. Thus, there are many reasons why it is interesting to investigate the impact of different forms of teacher education on teachers' capacities for teaching. Here, we wish to briefly discuss possible causal relations between

(1) the didactical organisations (particularly techniques) proposed for HTT1 and HTT2 by student teachers (close to graduation), and

(2) the context, contents and structure of their mathematics teacher education.

In Sec. 4 we have presented some data for (1); we will now say a little about (2) – at a fairly global level – and then discuss to what extent it may be seen as a cause for (1).

The formal assets of the education of lower secondary mathematics teachers are in several ways similar in Japan and France, when compared to Denmark. In Denmark, the whole training takes place in teacher training colleges, which are non-research institutions that are independent from universities. There, teachers are prepared to teach four different subjects, and the time allowed for the study of mathematics (integrated subject matter and didactics) is 0,7 years out of a total study time of 4 years (cf. Elle, 1996, 1999, for a more detailed description).

In Japan teacher education programme is 4 years, which are followed by a year of induction with reduced teaching load; in France the teacher education programme takes 5 years, but the final year comprises considerable teaching practice at a school, along with courses at the teacher education institute (cf. Pimm et al., 2003, chap. 5-6 for more information on teacher induction in Japan and France). In Japan, the induction year takes place after the students has graduated (and left) the university; otherwise it does bear some similarity to the fifth in France. But we chose our respondents to be around the end of the *official pre-service programme* (of 4 years in Japan, and 5 years in France). In both Japan and France, the training takes place in universities (or, for parts of the two final years in France, at teacher education institutes affiliated to the university). In both countries the studies begins with an extensive course program in "academic" mathematics (about 2 years in Japan, more than 3 years in France), followed by courses on pedagogy and didactics in the final years; and in both countries, mathematics teachers are prepared to teach just this one subject. Thus, while teachers in Japan and France receive university courses on advanced mathematics together with students preparing for other professions, Danish teacher students have a program of their own; as far as mathematics is concerned, it aims specifically and almost exclusively at a deepened knowledge of the subject as it is taught in school (in fact, primary as well as lower secondary level; there is no institutional distinction among these levels in Denmark, as a further contrast to the French and Japanese model).

In all three countries, there is some practice built into the program. It is usually spread throughout the four years of study in Denmark, while it is placed at the end of the French and Japanese programs. An overview of the space each element has in the study programs of the three countries is given in Fig. 5. Notice that "other" refers to the three other disciplines chosen by students in the Danish programme, while in the Japanese programme it refers courses meant to develop the general culture of the students (e.g. courses in English, history of education, citizenship etc.). We also note that in Denmark, a new teacher training programme is expected to be launched in 2007, and what is described here is the programme which our respondents had (almost) completed.

As mentioned in Sec.2, the techno-theoretical blocks of DOs – notably at the level of theory – differs considerably among countries and even institutions within a country. It seems to us that in all three countries, it is even difficult to recognise strong theoretical grounds among the students trained within a single institution. However, those tendencies we do see are quite likely to be caused from different contents in the element "didactics of mathematics" – such as occasional references to the theory of didactical situations (Brousseau, 1997) in France, to a recently developed system of mathematical competences in Denmark (Niss, 2002), and to the official curriculum and teaching aims in all three countries. The Japanese teacher students are particularly explicit in referring to national standards of mathematics instruction (cf. JSME, 2000) corresponding to the grade levels which are indicated in the tasks.

	Mathe-matics	**Didactics of mathematics**	**General ed. (pedagogy)**	**Teaching practice**	**Other**
DENMARK	0,7		0,7	0,6	2,0
FRANCE	3,7	0,4	0,3	0,6	-
JAPAN	2,2	0,2	0,4	0,2	1,0

Fig. 5. *Study time (in years) for different subjects in the lower secondary teacher education programmes of the three countries.*

When it comes to the DO techniques proposed by students (cf. Fig. 3), the tendency of Japanese and French pairs to attack the MO task with more appropriate methods, and to do so with the attention of *explaining* things to students, could well be seen as an effect of their rather extensive experience from university mathematics courses. The Danish pairs are fairly explicit about general principles they want to pursue: e.g. to make pupils work on more examples, for HTT1, or providing several explanations, as for HTT2. In the latter case, this explicitness occurs among 3 pairs in spite of a *de facto* lack of just one appropriate explanation. It could perhaps be linked to the relative extensive training in general pedagogy and to their broader subject matter horizon; indeed, their discussions involve more use of terms and ideas from general education.

It is well known that the degree of “deep understanding of fundamental mathematics” among experienced teachers does not depend directly on the amount of academic mathematics courses in their pre-service education (cf. in particular Ma, 1999, for a striking study in this direction). However this could to a large extent be explained by factors that do not affect our respondents, such as the conditions at schools for teachers’ continued intellectual development (cf. e.g. Stiegler and Hiebert, 1999). It may therefore not be seem surprising that the Danish pairs – with their much shorter training in mathematics – exhibit fewer (if any) appropriate didactical techniques related directly to techno-theoretical blocks of the MOs. However, we checked that both the product rule for negative integers and similarity of plane figures had been addressed during the courses in mathematics and didactics of mathematics which the 10 Danish students had followed.

In a sense, given the difference in terms of formal training in mathematics, it could look more surprising that *all* fifteen pairs seemed quite uncertain in dealing with the theoretical level of the MO related to HTT2; after all, the French students had almost 4 years of advanced mathematics courses, and both Japanese and French students have had courses on axiomatic algebra. But the punctual organisation of (negative) integer multiplication is not likely to be explicitly addressed in the last year of teacher training; and apparently, the students do not link it with the courses they had on algebra. Whether they will ever make such links would then depend on conditions that could help them do so after they start teaching, such as in-service training or collaboration with more experiences teachers.

6. Conclusion

The anthropological approach provided a theoretical framework to situate "teacher knowledge", corresponding to teachers tasks, with respect to mathematical knowledge which are aimed at in mathematics teaching and the corresponding types of tasks that their pupils work with. In particular, teachers' task types may refer to mathematical organisations which are to be worked on by the pupils, and rather precisely defined parts of the teachers' techniques to solve these tasks may then be studied through "simplified" tasks within a hypothetical context. Here, reference models of the related mathematical organisation are important to describe the didactical techniques.

Our empirical study, using two hypothetical teachers' tasks, show that certain systematic differences of the teacher education systems in France, Japan and Denmark can indeed explain some overall differences in the teacher students' performance on the tasks. Our method is based on a small number of carefully analysed conversations among pairs of teacher students about the two teachers' tasks; it presents itself as an alternative, even at larger scales, to questionnaire surveys which does not allow for spontaneous teacher interaction, and the present study could be merely seen as a first attempt to use and justify this method.

We note that even if with more data, we cannot infer directly from teacher students' performance on HTTs to the way students will eventually solve similar (or other) didactical tasks, and even less to the learning of their pupils. We can also not predict what results we would get with experienced mathematics teachers in the three countries as respondents; indeed, this would depend also on factors as induction and other in-service learning opportunities offered in each country. Our next step will be to extend our study to such respondents.

References

Alter, M & Pradl, G. (2006). How, Exactly, Are Teachers Best Prepared? *Education Week* **25** (41), 42-43.

Aristotle (1991) *Metaphysics*. Translation by J. McMahon. New York: Prometheus books. Original in Greek, probably around 330 B.C.

Ball, D. L. (1991) Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (ed.) *Advances in research on teaching*, vol. 2 (1-48). Greenwich CT: JAI Press.

Ball, D. & Bass, H (2000). Interweaving Content and Pedagogy in Teaching and Learning to Teach: Knowing and Using Mathematics. In J. Boaler (Ed). *Multiple Perspectives on Mathematics Teaching and Learning.* Westport: Ablex Publishing.

Barbé, J., Bosch, M., Espinoza, L. et Gascón, J. (2005): Didactic restrictions on teachers practice - the case of limits of functions at spanish high schools. *Educational Studies in Mathematics* **59** (1-3), 235-268.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dordrecht: Kluwer.

Bosch, M. and Gascón, J.: 2002, Organiser l'étude, 2. Théories et empiries. *Actes de la 11e école d'été de didactique des mathématiques*, 23-40. Grenoble : La Pensée Sauvage.

Chevallard, Y. : 1991, *La transposition didactique: Du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage, Editions.

Chevallard, Y. : 1999, L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques* 12(1), 73-112.

ECS, Education Commission of the States (2003).

Eight questions on teacher preparation: what does the research say? a summary of the findings. Denver, USA: ECS.

Elle, B. (1996), Teacher education in Denmark. In: T. Sander et al. (eds), *Teacher education in Europe. Evaluation and perspectives*, 69-99. National Reports prepared for a European Evaluation Conference, Universität Osnabrück, 1995.

Elle, B. (1999) Teacher education in Denmark – updating the SIGMA report of 1996. *TNTEE publications* **2** (2). Umeå: University of Umeå.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the learning of mathematics*, **11** (1), 26-32.

Henry, M. (2000) Evolution and Prospects of Preservice Secondary Mathematics Teacher Edcuation in France. *Journal of Mathematics Teacher Education* **3**, 171-179.

JSME (Japanese Society of Mathematics Education, 2000). *Mathematics program in Japan.* Tokyo.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah NJ: Erlbaum Publ.

Niss, M. and Højgaard-Jensen, T. (2002). *Kompetencer og matematiklæring: ideer og inspiration til udvikling af matematikundervisning i Danmark.* Copenhagen: Danish Ministry of Education.

E. Britton, L. Paine, D. Pimm (eds), *Comprehensive teacher induction*. Dordrecht: Kluwer.

Shimizu, Y. (1999) Aspects of mathematics teacher education in Japan: focusing on the teachers' roles. *Journal of Mathematics Teacher Education* **2**, 107-116.

Stiegler, J. & Hiebert, J. (1999), *The Teaching Gap*. New York: The Free Press.

Winsløw, C. & Durand-Guerrier, V. (2007). Education of lower secondary mathematics teachers in Denmark and France: A comparative study of characteristics of the systems and their products. *Nordic Studies in Mathematics education* vol. 12 no. 2, 5-32.

Appendix 1. Coding example.

Below is the outline of a session on HTT1 with the corresponding coding inserted. The students' real names are replaced by A and B.

A would like to ask the students how they found their results. B identifies first the additive (τ_{12}*), and then the multiplicative principle (τ_{11}*). A wants to have a discussion on these (τ_{16}*). They both clearly agree that the "ratio" (multiplication) is the right principle (τ_{10}*) and proceed to discuss how to get the students to understand that it is so. They want to find a "concrete activity" (with physical objects) but don't find one. B then suggests that using triangles with very different sides, one could produce more tasks that clearly shows the additive principle wrong; B shows a 1, 10, 10 triangle (drawn during separate preparation) to illustrate this idea (τ_{13}*, τ_{17}*). They discuss rather vague ideas of how this example could be related to some reality the students are familiar with. B suggests using overheads and measuring on the screen, using different distances between slide and projector, but they abandon the idea as "it would be difficult in practice". B then suggests using a computer to experiment with magnifications of the "map" [they mean the aerial photo of the task] "once it has been scanned in" (τ_{19a}*). Finally they discuss other aspects of geometry and maps that are only not related to the task.

Viviane Durand-Guerrier, Université de Montpellier 2, France, vdurand@math.univ-montp2.fr

Carl Winsløw, Université de Copenhague, Danemark, winslow@ind.ku.dk

Hiroaki Yoshida, Université d'Ochanomizu, Japan, yoshida@is.ocha.ac.jp