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Pazuki, Mehdi Fabien

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POLARIZED MORPHISMS BETWEEN ABELIAN VARIETIES

F. PAZUKI

Abstract. We study in this paper the Dynamical Manin-Mumford problem, focusing on the question of polarizability for endomorphisms of an abelian variety A and on the action of a Frobenius and its Verschiebung on the diagonal subvariety of $A \times A$.

1. Introduction

We study in this work the role of polarizability for morphisms between abelian varieties in the Dynamical Manin-Mumford problem. Our results give also a way to strengthen the article [8].

Let us recall a few definitions: an endomorphism $\psi : X \to X$ of a projective variety is said to have a *polarization* if there exists an ample divisor D on X such that $\psi^* D \sim dD$ for some $d > 1$, where $\sim$ stands for linear equivalence. Another (equivalent) way of defining this notion is to use an ample line bundle $\mathcal{L}$ over X such that $\psi^*\mathcal{L} = \mathcal{L}^{\otimes d}$. The integer d is called the *weight* of the morphism ψ.

A subvariety Y of X is *preperiodic* under ψ if there exists integers $m \geq 0$ and $s > 0$ such that $\psi^{m+s}(Y) = \psi^m(Y)$. We denote $\mathrm{Prep}_\psi(X)$ the set of preperiodic points of X under ψ. We will focus on the dynamical Manin-Mumford Conjecture 1.2.1 in [11]:

Conjecture 1.1. *(Algebraic Dynamical Manin-Mumford) Let $\psi : X \to X$ be an endomorphism of a projective variety over a number field k with a polarization, and let Y be a subvariety of X. If $Y \cap \mathrm{Prep}_\psi(X)$ is Zariski dense in Y, then Y is a preperiodic subvariety.*

This conjecture is very natural, it contains the classical Manin-Mumford conjecture proved by Raynaud in 1983: if one chooses $\psi = [n]$ and $X = A$ an abelian variety, then ψ is polarized with weight n^2 and the preperiodic points are just torsion points.

One can find some examples of non-trivial endomorphisms where Conjecture 1.1 is true for the diagonal subvariety of a power of an abelian variety. Let A be an abelian variety and $\mathcal{L}$ an ample symmetric line bundle. Consider the endomorphisms α and β on A^4 given by $\alpha(x, y, z, t) = (x + z, y + t, x - z, y - t)$ and $\beta(x, y, z, t) = (x + y, x - y, z + t, z - t)$. If we let $\mathcal{L}_4 = p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L} \otimes p_4^*\mathcal{L}$, then we have $\alpha^*\mathcal{L}_4 = \beta^*\mathcal{L}_4 = \mathcal{L}_4^{\otimes 2}$, so the morphism (α, β) is polarized by $\mathcal{L}_4$. Of course one has $\alpha^2 = \beta^2$, so the diagonal of $A^4 \times A^4$ is actually preperiodic under the morphism (α, β).

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One can consult [1] for another situation, outside the abelian realm, where this conjecture is proved, namely on $\mathbb{P}^1 \times \mathbb{P}^1$ under a coordinatewise polynomial action.

D. Ghioca and T. Tucker found a family of counterexamples to this conjecture in [4]. They use squares of elliptic curves with complex multiplication. This was followed by the paper [8] where counterexamples of dimension four were constructed over squares of CM abelian surfaces, using complex multiplication on curves of genus 2 and some polarizability lemma. We show in this work a more general theorem, valid in greater dimension as well. The main result obtained is Theorem 1.2, which gives a general way of producing dynamical systems with particular intersection properties.

Let k be a CM number field and $\bar{k}$ an algebraic closure. We will also denote the complex conjugation by a bar, but the context will always be clear. Let $\mathcal{O}_k$ be the ring of integers of k. Let A be an abelian variety over k with complex multiplication by k. Let S_1 be the set of ramified places of k, let S_2 be the set of places where A has bad reduction and let S_3 be the set of places $\mathfrak{a}$ such that $F_{\mathfrak{a}}^n \neq V_{\mathfrak{a}}^n$ for all integers n, where $F_{\mathfrak{a}}$ is the Frobenius and $V_{\mathfrak{a}}$ the Verschiebung. We will denote by $S = S_1 \cup S_2 \cup S_3$ the reunion of these places.

Theorem 1.2. *Let A be an abelian variety defined over a number field k that contains the field of complex multiplications $\mathrm{End}_{\bar{k}}(A) \otimes \mathbb{Q}$ and such that $\mathcal{O}_k \subset \mathrm{End}_{\bar{k}}(A)$. Let $\mathfrak{p} \notin S$. Let $F_{\mathfrak{p}}$ denote the Frobenius associated to $\mathfrak{p}$ and $V_{\mathfrak{p}}$ denote the Verschiebung associated to $F_{\mathfrak{p}}$. Then $F_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$ are polarizable and the dynamical system $(A \times A, F_{\mathfrak{p}} \times V_{\mathfrak{p}})$ together with the diagonal subvariety of $A \times A$ gives a counterexample to Conjecture 1.1.*

This statement shows that one can actually use any CM abelian variety to construct a counterexample. Moreover, the way of producing these examples explains the origin of the first ones, and in a more geometric way.

There is a new version of Conjecture 1.1 that can be found in [4], where one takes into account the action on the tangent space of $T_{X,x}$ at preperiodic points x. More precisely one has:

Conjecture 1.3. *(Ghioca, Tucker, Zhang) Let $\psi : X \to X$ be an endomorphism of a smooth projective variety over a number field k with a polarization, and let Y be a subvariety of X. Then Y is preperiodic under ψ if and only if there exists a Zariski dense subset of points $x \in Y \cap \mathrm{Prep}_{\psi}(X)$ such that the tangent subspace of Y at x is preperiodic under the induced action of ψ on the Grassmanian $\mathrm{Gr}_{\dim(Y)}(T_{X,x})$.*

Theorem 2.1 of [4] implies that Conjecture 1.3 holds when ψ is a group endomorphism. Hence our constructions provide examples of subvarieties Y containing a Zariski dense set of preperiodic points x but such that $T_{Y,x}$ is not preperiodic under the action of ψ.

Using Lattès maps, one can transport these constructions on $\mathbb{P}^1 \times \mathbb{P}^1$, hence, using the Segre embedding, also on $\mathbb{P}^3$. A good question for future work would then be: what is the situation on $\mathbb{P}^2$?

In the next section we give the proof of Theorem 1.2. Then, as previously stated in [8], [2] and [9], obtaining polarizability can be difficult and is linked to arithmetic properties of the field k. Thus, we complete this work by giving some explicit polarizability criterions for morphisms between abelian varieties.

2. Abelian varieties and Frobenius maps

Let us start with the following lemma:

Lemma 2.1. *Let A be a projective variety over a field k, of dimension g and let $\mathcal{L}$ be an ample line bundle. Let f, j, h be three endomorphisms of A such that $f \circ j = h$. Suppose $f^*\mathcal{L} = \mathcal{L}^{\otimes d}$ and $h^*\mathcal{L} = \mathcal{L}^{\otimes n}$. Then j is polarized by $\mathcal{M} = \mathcal{L}^{\otimes d}$, $d \mid n$ and $j^*\mathcal{M} = \mathcal{M}^{\otimes \frac{n}{d}}$.*

Proof. We calculate $(f \circ j)^*\mathcal{L} = j^*(f^*\mathcal{L}) = j^*(\mathcal{L}^{\otimes d})$, hence $j^*(\mathcal{L}^{\otimes d}) = \mathcal{L}^{\otimes n}$. Take the degree (associated to $\mathcal{L}$) to get: $\deg_{\mathcal{L}}(j) < \mathcal{L}^{\otimes d} >^g = < \mathcal{L}^{\otimes n} >^g$, which gives $\deg_{\mathcal{L}}(j) d^g < \mathcal{L} >^g = n^g < \mathcal{L} >^g$, which is equivalent, as $\mathcal{L}$ is ample, to $\deg_{\mathcal{L}}(j) d^g = n^g$. So there exists an integer m such that $n = dm$. This shows $j^*(\mathcal{L}^{\otimes d}) = (\mathcal{L}^{\otimes d})^{\otimes m}$. □

Now we will move on to the proof of Theorem 1.2.

Proof. One remark is that if k is large enough, S_2 is empty. One finds in [2] pages 16-17, Proposition 3.2-3.3, that if $\mathfrak{p} \notin S_1 \cup S_2$, then the Frobenius $F_{\mathfrak{p}}$ is polarized by a symmetric line bundle $\mathcal{L}$.

Let $V_{\mathfrak{p}}$ denote the Verschiebung associated to $F_{\mathfrak{p}}$. Let $N = \mathrm{Norm}(\mathfrak{p})$. We know that $F_{\mathfrak{p}}^*(\mathcal{L}) = \mathcal{L}^{\otimes N}$ and that $[N]^*\mathcal{L} = \mathcal{L}^{N^2}$. As we have $F_{\mathfrak{p}} \circ V_{\mathfrak{p}} = [N]$, we can apply Proposition 2.1 to get $V_{\mathfrak{p}}^*(\mathcal{L}^{\otimes N}) = (\mathcal{L}^{\otimes N})^{\otimes N} = \mathcal{L}^{\otimes N^2}$, hence $V_{\mathfrak{p}}$ is also polarizable, and with the same weight. Thus $F_{\mathfrak{p}} \times V_{\mathfrak{p}}$ is also polarized on $A \times A$.

Now, consider $\Delta = \{(P, P) \mid P \in A\}$, the diagonal subvariety of $A \times A$. This variety cannot be preperiodic under $\varphi = F_{\mathfrak{p}} \times V_{\mathfrak{p}}$, because it would imply $F_{\mathfrak{p}}^m = V_{\mathfrak{p}}^m$ for some positive integer m, which is impossible because it would imply $\mathfrak{p} \in S_3$. But $\Delta \cap \mathrm{Prep}_{\varphi}(A \times A)$ is dense in Δ because it contains all torsion points of Δ. □

Remark 2.2. This theorem is a way to generalize the previous counterexamples of [4] and [8]. For example when an elliptic curve E has a multiplication by $[i]$, the morphism $[2+i]$ corresponds in fact to the Frobenius $F_{(2-i)}$, as is shown in [10], Proposition 4.2 page 122.

Remark 2.3. This theorem gives as a by-product an explanation about the fact that one needs to avoid the ramified places if one searches for polarizability in the CM case. For the number field $\mathbb{Q}[i]$, the discriminant is -4, so as $(1+i) \mid (2)$, the morphism $[1+i]$ will not be polarizable. On the contrary, as $(2+i)$ and (2) are coprime ideals, $[2+i]$ is polarizable. This refines what is said in [9].

Remark 2.4. One can construct examples of polarized morphisms on $A \times A$ like in theorem 1.2 as soon as $\mathbb{Z} \subsetneq \mathrm{End}(A)$ and the Rosati involution is not trivial. We refer to [7] page 200, Theorem 2, for the classification of division algebras that can occur for $\mathrm{End}_{\mathbb{Q}}(A)$.

3. Polarizability criterions

A classical tool to get polarizability is the cube theorem. We refer to [8] for some formulas derived from this theorem and useful to get information on the weight of complex multiplications.

We give in this section other polarizability criterions, namely Proposition 3.1 for the action of the Rosati involution and Proposition 3.3 for the particular case of elliptic curves.

3.1. **Rosati involution.** Let A be an abelian variety over a field k and $\mathcal{L}$ be an ample line bundle. We denote by $\dagger$ the Rosati involution associated to $\mathcal{L}$. (See [6] page 137 for more details.) For any invertible line bundle $\mathcal{M}$, we let $\varphi_{\mathcal{M}}$ denote the classical application $a \mapsto t_a^*\mathcal{M} \otimes \mathcal{M}^{-1}$ from A to $\mathrm{Pic}^0(A)$.

Proposition 3.1. *Let A be an abelian variety, ψ an endomorphism of A and $\mathcal{L}$ an ample line bundle. Then ψ is polarized by $\mathcal{L}$ with weight d if and only if one has $\psi^\dagger\psi = [d]$, where $\dagger$ is associated to $\mathcal{L}$.*

Proof. Let $\mathrm{NS}_{\mathbb{Q}}(A)$ be the Néron-Severi group of A tensored by $\mathbb{Q}$. Let $\mathrm{End}_{\mathbb{Q}}^{\dagger}(A)$ be the group of endomorphism fixed by $\dagger$. We thus have an isomorphism (see [6] page 137)

$$\begin{aligned} H : \mathrm{NS}_{\mathbb{Q}}(A) &\longrightarrow \mathrm{End}_{\mathbb{Q}}^{\dagger}(A) \\ \mathcal{M} &\mapsto \varphi_{\mathcal{L}}^{-1} \circ \varphi_{\mathcal{M}}. \end{aligned}$$

We then calculate that $H(\psi^*\mathcal{L}) = \psi^\dagger\psi$ and $H(\mathcal{L}^{\otimes d}) = [d]$, so we can express the polarizability condition $\psi^*\mathcal{L} = \mathcal{L}^{\otimes d}$ by $\psi^\dagger\psi = [d]$. □

Remark 3.2. If A has complex multiplication, then one has $\psi^\dagger = \overline{\psi}$. Hence, to study Conjecture 1.1, as we want to find endomorphisms ψ such that for all integers $m \geq 1$, $\psi^m \neq (\psi^\dagger)^m$, in the CM case it boils down to finding a number α such that $\alpha\overline{\alpha} \in \mathbb{Z}$, $\alpha \notin \mathbb{Z}$ and $\alpha/\overline{\alpha}$ is not a root of unity.

Application: *back to multiplication by* $1+\zeta_5$ *not polarized by* Θ. Thanks to this Rosati action, we can add a remark to one of the examples of [8] where the morphism $[1+\zeta_5]$ is not polarized by the divisor Θ on the jacobian of a genus 2 curve. But now a little calculation in the particular example of $z = 4 + 3\zeta_5 + 12\zeta_5^2$ gives $z\overline{z} = 121$ and $z/\overline{z}$ is not a root of unity. Hence by Proposition 3.1 we get that $[z]$ is polarized by Θ.

3.2. **Elliptic curves.** In the particular case of elliptic curves, one can get a precise condition for polarizability. This is due to the fact that the variety is simple on the one hand and the fact that the divisor support is just a point on the other hand.

Proposition 3.3. *Let E be an elliptic curve over a field k of characteristic zero and f be an isogeny of E of degree d. We denote by $E[2]$ the group of 2-torsion points over $\bar{k}$. Then we have*

$$\Big(\mathrm{Card}(E[2] \cap \mathrm{Ker}(f)) \neq 2\Big) \Leftrightarrow \Big(f^*(O) \sim d(O)\Big).$$

Proof. Let $H = E[2] \cap \mathrm{Ker}(f)$ and $G = \mathrm{Ker}(f) \backslash H$. Let $m = \mathrm{Card}(H)$. We know that $m \in \{1, 2, 4\}$. Let us choose any short Weierstrass model $y^2 = x^3 + ax + b$. We calculate:

$$f^*(O) = \sum_{P \in \mathrm{Ker}(f)} (P) = \sum_{P \in H} (P) + \sum_{P \in G} (P) = \sum_{P \in H} (P) + \sum_{P \in G/\pm 1} ((P) + (-P)),$$

and as $\mathrm{div}(x - x(P)) = (P) + (-P) - 2(O)$, we have

$$\sum_{P \in G/\pm 1} ((P) + (-P)) \sim 2\Big(\frac{d-m}{2}\Big)(O),$$

thus $f^*(O) \sim \sum_{P \in H} (P) + (d - m)(O)$. Then we have three cases:

- if $m = 1$, then $H = \{O\}$ and $f^*(O) \sim (O) + (d-1)(O) \sim d(O)$,
- if $m = 2$, then $H = \{O, P\}$ and $f^*(O) \sim (O) + (P) + (d-2)(O) \sim (P) + (d-1)(O)$, then the divisor $D = (P) - (O)$ is of degree 0 but is not the divisor of a rational function on E because $P - O \neq O$,
- if $m = 4$, then $H = \{O, P_1, P_2, P_1 + P_2\}$ and $f^*(O) \sim (O) + (P_1) + (P_2) + (P_1+P_2) + (d-4)(O) \sim d(O)$ because $\operatorname{div}(y) = (P_1) + (P_2) + (P_1+P_2) - 3(O)$.

□

Remark 3.4. Let E be the elliptic curve with affine model $y^2 = x^3 + x$. It has complex multiplication by $[i] : (x, y) \to (-x, iy)$. Then the isogeny $[1+i]$ has degree 2, and as $[2] = [1+i][1-i]$, we get $\operatorname{Ker}[1+i] \subset E[2]$, hence $\operatorname{Card}(E[2] \cap \operatorname{Ker}[1+i]) = 2$, hence by Proposition 3.3 the morphism $[1+i]$ is not polarized. But as shown in [8], the morphism $[2+i]$ is polarized, another proof of this fact is that $[5] = [2+i][2-i]$, hence $\operatorname{Ker}[2+i] \subset E[5]$, but $E[5] \cap E[2] = O$, hence again by Proposition 3.3 the morphism $[2+i]$ is polarized, with weight 5.

Remark 3.5. This result sharpens the remark made in the introduction of [9] concerning elliptic curves. See the remark 2.3 below for more details.

3.3. **Explicit Lattès dynamical system.** Let E be the elliptic curve with affine model $y^2 = x^3 + x$. It has complex multiplication by $[i] : (x, y) \to (-x, iy)$. Let $\pi : E \to \mathbb{P}^1$ be defined by $\pi(x, y) = x$ and $\pi(O) = \infty$. Then we get the following picture:

$$\begin{tikzcd}
E \arrow[r, "{[2+i]}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\
\mathbb{P}^1 \arrow[r, "\varphi"'] & \mathbb{P}^1
\end{tikzcd}$$

A direct calculation shows that the Lattès map φ can be expressed on the affine chart as

$$\varphi(x) = \frac{(3-4i)x(x^2+1-2i)^2}{(5x^2+1+2i)^2}.$$

Remark 3.6. We thus get the x-coordinate of the four non-trivial $[2+i]$-torsion points: $\pm x = \sqrt{\frac{\sqrt{5}-1}{10}} + i\,\frac{1}{5}\sqrt{\frac{10}{\sqrt{5}-1}}$.

For the Lattès map of $[2-i]$, we get on the affine chart

$$\psi(x) = \frac{(3+4i)x(x^2+1+2i)^2}{(5x^2+1-2i)^2}.$$

Now take a look at

$$\begin{aligned}
\delta : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\
(s, t) &\longmapsto (\varphi(s), \psi(t)).
\end{aligned}$$

Then if $D = \{\infty\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}$, we get $\delta^* D \sim 5D$, hence δ is polarized by D with weight 5. This gives an explicit counterexample to Conjecture 1.1 in the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$ as in the constructions of [4]. See [3] for Lattès dynamical systems.

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IMB, Univ. Bordeaux 1, 351, cours de la Libération, 33405 Talence, France,
E-mail address: `fabien.pazuki@math.u-bordeaux1.fr`