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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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Abstract. In this paper we prove that for any associative ring \( R \), and for any left \( R \)-module \( M \) with finite projective dimension, the Gorenstein injective dimension \( \text{Gid}_R M \) equals the usual injective dimension \( \text{id}_R M \). In particular, if \( \text{Gid}_R R \) is finite, then also \( \text{id}_R R \) is finite, and thus \( R \) is Gorenstein (provided that \( R \) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \((R, \mathfrak{m}, k)\), the Gorenstein rings are characterized by the condition \( \text{id}_R R < \infty \). From the dual of [10] Proposition (2.27) ([6] Proposition 10.2.3] is a special case) it follows that the Gorenstein injective dimension \( \text{Gid}_R (–) \) is a refinement of the usual injective dimension \( \text{id}_R (–) \) in the following sense:

For any \( R \)-module \( M \) there is an inequality \( \text{Gid}_R M \leq \text{id}_R M \), and if \( \text{id}_R M < \infty \), then there is an equality \( \text{Gid}_R M = \text{id}_R M \).

Now, since the injective dimension \( \text{id}_R R \) of \( R \) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \( \text{Gid}_R R \) of \( R \) measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring \( R \) with \( \text{Gid}_R R < \infty \) also has \( \text{id}_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

This result is proved by Christensen [2] Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that \((R, \mathfrak{m}, k)\) is a commutative local Noetherian ring, and let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}^m_R(k, M) \neq 0 \) for some \( m \in \mathbb{N}_0 \) (this happens for example if \( M \neq 0 \) is finitely generated). If either

(i) \( \text{Gid}_R M < \infty \) and \( \text{id}_R M < \infty \) or
(ii) \( \text{fd}_R M < \infty \) and \( \text{Gid}_R M < \infty \),

then \( R \) is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem [3.2] itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_{R}M\), \(\text{id}_{R}M\), and \(\text{id}_{R}M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_{R}M\), \(\text{Gid}_{R}M\), and \(\text{Gid}_{R}M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_{R}M < \infty\), then \(\text{Gid}_{R}M = \text{id}_{R}M\). In particular, if \(\text{Gid}_{R}R < \infty\), then also \(\text{id}_{R}R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_{R}M \leq \text{id}_{R}M\) always, it suffices to prove that \(\text{id}_{R}M \leq \text{Gid}_{R}M\). Naturally, we may assume that \(\text{Gid}_{R}M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_{R}M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \rightarrow E_{1} \rightarrow E_{0} \rightarrow E_{-1} \rightarrow \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_{R}(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \text{Ker}(E_{1} \rightarrow E_{0})\). In particular, there exists a short exact sequence \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_{R}M < \infty\), it follows by [11, Lemma 1.3] that \(\text{Ext}_{R}^{1}(M, M') = 0\). Thus \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_{R}M > 0\). By [10, Theorem (2.15)] there exists an exact sequence \(0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0\) where \(H\) is Gorenstein injective and \(\text{id}_{R}C = \text{Gid}_{R}M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \rightarrow H' \rightarrow I \rightarrow H \rightarrow 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0
\]

\[
0 \rightarrow P \rightarrow I \rightarrow C \rightarrow 0
\]

\[
H' \rightarrow H'
\]

Since \( I \) is injective and \( \text{id}_{\mathbb{R}}C = \text{Gid}_{\mathbb{R}}M - 1 \) we get \( \text{id}_{\mathbb{R}}P \leq \text{Gid}_{\mathbb{R}}M \) by the second row. Since \( H' \) is Gorenstein injective and \( \text{pd}_{\mathbb{R}}M < \infty \), it follows (as before) by [4, Lemma 1.3] that \( \text{Ext}^1_{\mathbb{R}}(M, H') = 0 \). Consequently, the first column \( 0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0 \) splits. Therefore \( P \cong M \oplus H' \), and hence \( \text{id}_{\mathbb{R}}M \leq \text{id}_{\mathbb{R}}P \leq \text{Gid}_{\mathbb{R}}M \). \( \square \)

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \( M \) is an \( \mathbb{R} \)-module with \( \text{id}_{\mathbb{R}}M < \infty \), then \( \text{Gpd}_{\mathbb{R}}M = \text{pd}_{\mathbb{R}}M \). \( \square \)

Theorem (2.6) below is a “flat version” of the two previous theorems. First recall the following.

**Definition 2.3.** The **left finitistic projective dimension** \( \text{LeftFPD}(\mathbb{R}) \) of \( \mathbb{R} \) is defined as

\[
\text{LeftFPD}(\mathbb{R}) = \sup \{ \text{pd}_{\mathbb{R}}M \mid M \text{ is a left } \mathbb{R}\text{-module with } \text{pd}_{\mathbb{R}}M < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(\mathbb{R}) \) of \( \mathbb{R} \) is defined similarly.

**Remark 2.4.** When \( \mathbb{R} \) is commutative and Noetherian, we have that \( \text{LeftFPD}(\mathbb{R}) \) and \( \text{RightFPD}(\mathbb{R}) \) equals the Krull dimension of \( \mathbb{R} \), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \( \mathbb{R} \)-module \( M \) the inequality

\[
\text{Gid}_{\mathbb{R}}\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_{\mathbb{R}}M
\]

holds. If \( \mathbb{R} \) is right coherent, then we have \( \text{Gid}_{\mathbb{R}}\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_{\mathbb{R}}M \). \( \square \)

We are now ready to state:

**Theorem 2.6.** For any \( \mathbb{R} \)-module \( M \), the following conclusions hold:

(i) Assume that \( \text{LeftFPD}(\mathbb{R}) \) is finite. If \( \text{fd}_{\mathbb{R}}M < \infty \), then \( \text{Gid}_{\mathbb{R}}M = \text{id}_{\mathbb{R}}M \).

(ii) Assume that \( \mathbb{R} \) is left and right coherent with finite \( \text{RightFPD}(\mathbb{R}) \). If \( \text{id}_{\mathbb{R}}M < \infty \), then \( \text{Gfd}_{\mathbb{R}}M = \text{fd}_{\mathbb{R}}M \).

**Proof.** (i) If \( \text{fd}_{\mathbb{R}}M < \infty \), then also \( \text{pd}_{\mathbb{R}}M < \infty \), by [11, Proposition 6] (since \( \text{LeftFPD}(\mathbb{R}) < \infty \)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \( \mathbb{R} \) is left coherent, we have that \( \text{fd}_{\mathbb{R}}\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_{\mathbb{R}}M < \infty \), by [12, Lemma 3.1.4]. By assumption, \( \text{RightFPD}(\mathbb{R}) < \infty \), and therefore also
pd_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [11, Proposition 6]. Now Theorem 2.11 gives that \text{Gid}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$. It is well known that \text{fd}_RM = \text{id}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) (without assumptions on \( R \)), and by Proposition 2.5 above, we also get \text{Gfd}_R\text{M} = \text{Gid}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$, since \( R \) is right coherent. The proof is done. \qed

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that \( R \) is commutative and Noetherian. For an \( R \)-module \( M \), an integer \( n \), and a prime ideal \( p \) in \( R \), we write \( \beta^R_n(p, M) \), respectively, \( \mu^R_n(p, M) \), for the \( n \)th Betti number, respectively, \( n \)th Bass number, of \( M \) at \( p \).

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an \( R \)-module \( M \) to be the set

\[ \text{supp}_R M = \{ p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_0 : \beta^R_n(p, M) \neq 0 \}. \]

Let us mention the most basic results about the small support, all of which can be found in [8, pp. 157–159] and [7, Chapter 14]:

(a) The small support, \( \text{supp}_R M \), is contained in the usual (large) support, \( \text{Supp}_R M \), and \( \text{supp}_R M = \text{Supp}_R M \) if \( M \) is finitely generated. Also, if \( M \neq 0 \), then \( \text{supp}_R M \neq \emptyset \).

(b) \( \text{supp}_R M = \{ p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_0 : \mu^R_n(p, M) \neq 0 \} \).

(c) Assume that \((R, \mathfrak{m}, k)\) is local. If \( M \) is an \( R \)-module with finite depth, that is, \( \text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}^m_R(k, M) \neq 0 \} < \infty \) (this happens for example if \( M \neq 0 \) is finitely generated), then \( m \in \text{supp}_R M \), by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

**Theorem 3.2.** Assume that \( R \) is commutative and Noetherian. Let \( M \) be any \( R \)-module, and assume that any of the following four conditions is satisfied:

(i) \( \text{Gpd}_R M < \infty \) and \( \text{id}_R M < \infty \),

(ii) \( \text{pd}_R M < \infty \) and \( \text{Gid}_R M < \infty \),

(iii) \( R \) has finite Krull dimension, and \( \text{Gfd}_R M < \infty \) and \( \text{id}_R M < \infty \),

(iv) \( R \) has finite Krull dimension, and \( \text{fd}_R M < \infty \) and \( \text{id}_R M < \infty \).

Then \( R_p \) is a Gorenstein local ring for all \( p \in \text{supp}_R M \). \qed

**Corollary 3.3.** Assume that \((R, \mathfrak{m}, k)\) is a commutative local Noetherian ring. If there exists an \( R \)-module \( M \) of finite depth, that is, \( \text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}^m_R(k, M) \neq 0 \} < \infty \),

and which satisfies either

(i) \( \text{Gfd}_R M < \infty \) and \( \text{id}_R M < \infty \), or

(ii) \( \text{fd}_R M < \infty \) and \( \text{Gid}_R M < \infty \),

then \( R \) is Gorenstein. \qed
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References


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