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Holm, Henrik Granau

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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

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Abstract. In this paper we prove that for any associative ring $R$, and for any left $R$-module $M$ with finite projective dimension, the Gorenstein injective dimension $Gid_R M$ equals the usual injective dimension $id_R M$. In particular, if $Gid_R R$ is finite, then also $id_R R$ is finite, and thus $R$ is Gorenstein (provided that $R$ is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings $(R; m; k)$, the Gorenstein rings are characterized by the condition $id_R R < \infty$. From the dual of [10] Proposition (2.27) ([6] Proposition 10.2.3] is a special case) it follows that the Gorenstein injective dimension $Gid_R(-)$ is a refinement of the usual injective dimension $id_R(-)$ in the following sense:

For any $R$-module $M$ there is an inequality $Gid_R M \leq id_R M$, and if $id_R M < \infty$, then there is an equality $Gid_R M = id_R M$.

Now, since the injective dimension $id_R R$ of $R$ measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $Gid_R R$ of $R$ measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring $R$ with $Gid_R R < \infty$ also has $id_R R < \infty$ (and hence $R$ is Gorenstein, provided that $R$ is commutative and Noetherian).

This result is proved by Christensen [2] Theorem (6.3.2)] in the case where $(R; m; k)$ is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that $(R; m; k)$ is a commutative local Noetherian ring, and let $M$ be an $R$-module of finite depth, that is, $Ext^m_R(k; M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

$(i)$ $Gid_R M < \infty$ and $id_R M < \infty$ or

$(ii)$ $fd_R M < \infty$ and $Gid_R M < \infty$,

then $R$ is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, m, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_R M\), \(\text{id}_R M\), and \(\text{id}_R M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_R M\), \(\text{Gfd}_R M\), and \(\text{Gid}_R M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\). In particular, if \(\text{Gid}_R R < \infty\), then also \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_R M \leq \text{id}_R M\) always, it suffices to prove that \(\text{id}_R M \leq \text{Gid}_R M\). Naturally, we may assume that \(\text{Gid}_R M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_R M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \text{Ker}(E_1 \to E_0)\). In particular, there exists a short exact sequence \(0 \to M' \to E \to M \to 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows by [4] Lemma 1.3 that \(\text{Ext}^1_R(M, M') = 0\). Thus \(0 \to M' \to E \to M \to 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_R M > 0\). By [10] Theorem (2.15) there exists an exact sequence \(0 \to M \to H \to C \to 0\) where \(H\) is Gorenstein injective and \(\text{id}_R C = \text{Gid}_R M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \to H' \to I \to H \to 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{cccccccc}
0 & 0 & \rightarrow & M & \rightarrow & H & \rightarrow & C & \rightarrow & 0 \\
0 & \rightarrow & P & \rightarrow & I & \rightarrow & C & \rightarrow & 0 \\
& \rightarrow & H' & \rightarrow & H' & & & & \\
& 0 & \rightarrow & 0 & & & & & \\
\end{array}
\]

Since \( I \) is injective and \( \text{id}_R \mathcal{M} \leq \text{Gid}_R \mathcal{M} \) by the second row. Since \( H' \) is Gorenstein injective and \( \text{pd}_R \mathcal{M} < \infty \), it follows (as before) by [1, Lemma 1.3] that \( \text{Ext}_R^1(\mathcal{M}, H') = 0 \). Consequently, the first column \( 0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0 \) splits. Therefore \( P \cong M \otimes H' \), and hence \( \text{id}_R \mathcal{M} \leq \text{Gid}_R \mathcal{M} \).

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \( \mathcal{M} \) is an \( R \)-module with \( \text{id}_R \mathcal{M} < \infty \), then \( \text{Gpd}_R \mathcal{M} = \text{pd}_R \mathcal{M} \). \( \square \)

Theorem (2.6) below is a “flat version” of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \( \text{LeftFPD}(R) \) of \( R \) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R \mathcal{M} \mid \mathcal{M} \text{ is a left } R\text{-module with } \text{pd}_R \mathcal{M} < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(R) \) of \( R \) is defined similarly.

**Remark 2.4.** When \( R \) is commutative and Noetherian, we have that \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) equals the Krull dimension of \( R \), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \( R \)-module \( \mathcal{M} \) the inequality

\[
\text{Gid}_R \text{Hom}_\mathbb{Z}(\mathcal{M}, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R \mathcal{M}
\]

holds. If \( R \) is right coherent, then we have \( \text{Gid}_R \text{Hom}_\mathbb{Z}(\mathcal{M}, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R \mathcal{M} \). \( \square \)

We are now ready to state:

**Theorem 2.6.** For any \( R \)-module \( \mathcal{M} \), the following conclusions hold:

(i) Assume that \( \text{LeftFPD}(R) \) is finite. If \( \text{fd}_R \mathcal{M} < \infty \), then \( \text{Gid}_R \mathcal{M} = \text{id}_R \mathcal{M} \).

(ii) Assume that \( R \) is left and right coherent with finite \( \text{RightFPD}(R) \). If \( \text{id}_R \mathcal{M} < \infty \), then \( \text{Gfd}_R \mathcal{M} = \text{fd}_R \mathcal{M} \).

**Proof.** (i) If \( \text{fd}_R \mathcal{M} < \infty \), then also \( \text{pd}_R \mathcal{M} < \infty \), by [11, Proposition 6] (since \( \text{LeftFPD}(R) < \infty \)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \( R \) is left coherent, we have that \( \text{fd}_R \text{Hom}_\mathbb{Z}(\mathcal{M}, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R \mathcal{M} < \infty \), by [12, Lemma 3.1.4]. By assumption, \( \text{RightFPD}(R) < \infty \), and therefore also
pd_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \leq \infty$, by [11, Proposition 6]. Now Theorem 2.1 gives that 
\text{Gid}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}). It is well known that
\[ \text{fd}_RM = \text{id}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \]
(without assumptions on $R$), and by Proposition 2.5 above, we also get $\text{Gfd}_RM = \text{Gid}_R\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$, since $R$ is right coherent. The proof is done. 

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that $R$ is commutative and Noetherian. For an $R$-module $M$, an integer $n$, and a prime ideal $p$ in $R$, we write $\beta_n^R(p, M)$, respectively, $\mu_n^R(p, M)$, for the $n$th Betti number, respectively, $n$th Bass number, of $M$ at $p$.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an $R$-module $M$ to be the set

\[ \text{supp}_RM = \{ p \in \text{Spec }R \mid \exists n \in \mathbb{N}_0 : \beta_n^R(p, M) \neq 0 \} . \]

Let us mention the most basic results about the small support, all of which can be found in [8, pp. 157–159] and [7, Chapter 14]:

(a) The small support, $\text{supp}_RM$, is contained in the usual (large) support, $\text{Supp}_RM$, and $\text{Supp}_RM = \text{Supp}_RM$ if $M$ is finitely generated. Also, if $M \neq 0$, then $\text{supp}_RM \neq 0$.

(b) $\text{supp}_RM = \{ p \in \text{Spec }R \mid \exists n \in \mathbb{N}_0 : \mu_n^R(p, M) \neq 0 \} .

(c) Assume that $(R, m, k)$ is local. If $M$ is an $R$-module with finite depth, that is,

\[ \text{depth}_RM := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty \]

(this happens for example if $M \neq 0$ is finitely generated), then $m \in \text{supp}_RM$, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

**Theorem 3.2.** Assume that $R$ is commutative and Noetherian. Let $M$ be any $R$-module, and assume that any of the following four conditions is satisfied:

(i) $\text{Gpd}_RM < \infty$ and $\text{id}_RM < \infty$,

(ii) $\text{pd}_RM < \infty$ and $\text{Gid}_RM < \infty$,

(iii) $R$ has finite Krull dimension, and $\text{Gfd}_RM < \infty$ and $\text{id}_RM < \infty$,

(iv) $R$ has finite Krull dimension, and $\text{fd}_RM < \infty$ and $\text{Gid}_RM < \infty$.

Then $R_p$ is a Gorenstein local ring for all $p \in \text{supp}_RM$. 

**Corollary 3.3.** Assume that $(R, m, k)$ is a commutative local Noetherian ring. If there exists an $R$-module $M$ of finite depth, that is,

\[ \text{depth}_RM := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty, \]

and which satisfies either

(i) $\text{Gfd}_RM < \infty$ and $\text{id}_RM < \infty$, or

(ii) $\text{fd}_RM < \infty$ and $\text{Gid}_RM < \infty$,

then $R$ is Gorenstein.
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References


Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, Danmark
E-mail address: holm@math.ku.dk