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Holm, Henrik Granau

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GORENSTEIN DERIVED FUNCTORS

HENRIK HOLM

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(\cdot, \cdot)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}^n_R(\cdot, \cdot)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $\otimes_R$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.1:2)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).

- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.

- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring \( R \), the usual right derived functors \( \text{Ext}^n_R(\cdot, \cdot) \) of \( \text{Hom}_R(\cdot, \cdot) \) are important in homological studies of \( R \). The material presented here deals with the Gorenstein right derived functors \( \text{Ext}^n_{\mathcal{GP}}(\cdot, \cdot) \) and \( \text{Ext}^n_{\mathcal{GI}}(\cdot, \cdot) \) of \( \text{Hom}_R(\cdot, \cdot) \).

More precisely, let \( N \) be a fixed \( R \)-module. For an \( R \)-module \( M \) that has a proper left \( \mathcal{GP} \)-resolution \( G = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0 \) (please see [2.1] below for the definition of proper resolutions), we define

\[
\text{Ext}^n_{\mathcal{GP}}(M, N) := H^n(\text{Hom}_R(G, N)).
\]

From [2.4] it will follow that \( \text{Ext}^n_{\mathcal{GP}}(\cdot, \cdot) \) is a well-defined contravariant functor, defined on the full subcategory, \( \text{LeftRes}_R(\mathcal{GP}) \), of \( \mathcal{M} \), consisting of all \( R \)-modules that have a proper left \( \mathcal{GP} \)-resolution.

For a fixed \( R \)-module \( M' \) there is a similar definition of the functor \( \text{Ext}^n_{\mathcal{GI}}(M', \cdot) \), which is defined on the full subcategory, \( \text{RightRes}_R(\mathcal{GI}) \), of \( \mathcal{M} \), consisting of all \( R \)-modules that have a proper right \( \mathcal{GI} \)-resolution. Now, the best one could hope for is the existence of isomorphisms,

\[
\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N),
\]

which are functorial in each variable \( M \in \text{LeftRes}_R(\mathcal{GP}) \) and \( N \in \text{RightRes}_R(\mathcal{GI}) \).

The aim of this paper is to show a slightly weaker result.

When \( R \) is \( n \)-Gorenstein (meaning that \( R \) is both left and right Noetherian, with self-injective dimension \( \leq n \) from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms \( \text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N) \) for all \( R \)-modules \( M \) and \( N \).

It is important to note that for an \( n \)-Gorenstein ring \( R \), we have \( \text{Gpd}_R M < \infty \), \( \text{Gid}_R M < \infty \), and also \( \text{Gfpd}_R M < \infty \) for all \( R \)-modules \( M \); please see [10] Theorems 11.2.1, 11.5.1, 11.7.6. For any ring \( R \), [12] Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category \( \text{LeftRes}_R(\mathcal{GP}) \) contains all \( R \)-modules \( M \) with \( \text{Gpd}_R M < \infty \); that is, every \( R \)-module with finite \( G \)-projective dimension has a proper left \( \mathcal{GP} \)-resolution. Also, every \( R \)-module with finite \( G \)-injective dimension has a proper right \( \mathcal{GI} \)-resolution. So \( \text{RightRes}_R(\mathcal{GI}) \) contains all \( R \)-modules \( N \) with \( \text{Gid}_R N < \infty \).

Theorem 5.6 in this text proves that the functorial isomorphisms \( \text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N) \) hold over arbitrary rings \( R \), provided that \( \text{Gpd}_R M < \infty \) and \( \text{Gid}_R N < \infty \). By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product \( - \otimes_R - \), using proper left \( \mathcal{GP} \)-resolutions and proper left \( \mathcal{GF} \)-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2] in the case when \( R \) is \( n \)-Gorenstein.

2. Preliminaries

Let \( T : \mathcal{C} \rightarrow \mathcal{E} \) be any additive functor between abelian categories. One usually derives \( T \) using resolutions consisting of projective or injective objects (if the category \( \mathcal{C} \) has enough projectives or injectives). This section is a very brief note on how to derive functors \( T \) with resolutions consisting of objects in some subcategory \( \mathcal{X} \subseteq \mathcal{C} \). The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $X^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence
\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \to \text{Hom}_\mathcal{C}(X, X_1) \to \text{Hom}_\mathcal{C}(X, X_0) \to \text{Hom}_\mathcal{C}(X, M) \to 0
\]
is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_\mathcal{C}(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let $f: M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram
\[
\cdots \to X_2 \to X_1 \to X_0 \to M \to 0 \\
\downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0 \quad \downarrow f \\
\cdots \to X_2' \to X_1' \to X_0' \to M' \to 0
\]
where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \to X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precovers of $M \in \mathcal{C}$ is a morphism $\varphi: X \to M$, where $X \in \mathcal{X}$, such that the sequence
\[
\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', X) \to \text{Hom}_\mathcal{C}(X', M) \to 0
\]
is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \to \text{Ker}(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor
\[
L_n^\mathcal{X}T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L_n^X T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_C(\mathcal{X})$. Similarly, the $n^{th}$ right derived functor

$$R_n^Y T : \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R_n^Y T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_C(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition \ref{prop:rightderived}.

The situation where $T$ is contravariant is handled similarly. We refer to \cite{9, Section 8.2} for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in \cite{2.4}

$$R_n^X F(-, N) : \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R_n^Y F(M, -) : \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.$$  

If furthermore $M \in \text{LeftRes}_C(\mathcal{X})$ and $N \in \text{RightRes}_D(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R_n^X F(M, N) \cong R_n^Y F(M, N),$$

functorial in $M$ and $N$. Here we wrote $R_n^X F(M, N)$ for the functor $R_n^X F(-, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R_n^X F(-, N)[M].$$

Enochs and Jenda have in \cite{5} developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult \cite{5, Definition 2.1}). Instead we shall focus on the following result:

\textbf{Theorem 2.6.} Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_C(\mathcal{X})$ and $\text{RightRes}_D(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{Y}$.

(ii) Every $M \in \mathcal{X}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \mathcal{Y}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R_n^X F(M, N) \cong R_n^Y F(M, N),$$

for all $M \in \mathcal{X}$ and $N \in \mathcal{Y}$.
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [5, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories $\mathcal{X}$, $\tilde{\mathcal{X}}$, $\mathcal{C}$ and $\mathcal{Y}$, $\tilde{\mathcal{Y}}$, $\mathcal{D}$, including the categories mentioned in [1.1] For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules, 

$$P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \to P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-,-)$

We now return to categories of modules. We use $\widehat{\mathcal{GP}}, \widehat{\mathcal{GI}}$ and $\widehat{\mathcal{GF}}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\mathcal{GP}$-precovers are always surjective, and $\widehat{\mathcal{GP}}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-,-) : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} = \mathcal{GP}, \quad \mathcal{Y} = \mathcal{GI}, \quad \mathcal{Y} = \mathcal{GF}.$$  

In this case we define, in the sense of section 2.4

$$\text{Ext}_{\mathcal{GP}}^n(-,-) = R_{\mathcal{GP}}^n\text{Hom}_R(-,-) \quad \text{and} \quad \text{Ext}_{\mathcal{GI}}^n(M,-) = R_{\mathcal{GI}}^n\text{Hom}_R(M,-),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $\mathcal{GP}$-precover of $M$ (please see Remark 2.13), and $\text{pd}_R K = \text{Gpd}_R M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $\mathcal{GP}$-resolution (that is, there is an inclusion $\widehat{\mathcal{GP}} \subseteq \text{LeftRes}_M(\mathcal{GP})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_R M < \infty$. Then

$$\text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an \( R \)-module \( M \) is given by

\[
\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \}.
\]

It also follows that if \( \text{pd}_R M < \infty \), then every projective resolution of \( M \) is actually a proper left \( \mathcal{GP} \)-resolution of \( M \).

Lemma 3.4. Assume that \( M \) is an \( \mathcal{R} \)-module with finite Gorenstein projective dimension, and let \( \mathcal{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0 \) be an augmented proper left \( \mathcal{GP} \)-resolution of \( M \) (which exists by Proposition 3.1). Then \( \text{Hom}_R(\mathcal{G}^+, H) \) is exact for all Gorenstein injective modules \( H \).

Proof. We split the proper resolution \( \mathcal{G}^+ \) into short exact sequences. Hence it suffices to show exactness of \( \text{Hom}_R(\mathcal{S}, H) \) for all Gorenstein injective modules \( H \) and all short exact sequences

\[
\mathcal{S} = 0 \to K \to G \to M \to 0,
\]

where \( G \to M \) is a \( \mathcal{GP} \)-precover of some module \( M \) with \( \text{Gpd}_R M < \infty \) (recall that \( \mathcal{GP} \)-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

\[
\mathcal{S}_0 = 0 \to K_0 \to G_0 \to M \to 0,
\]

where \( G_0 \to M \) is a \( \mathcal{GP} \)-precover and \( \text{pd}_R K_0 < 1 \).

It is easy to see (as in Proposition 2.2) that the complexes \( \mathcal{S} \) and \( \mathcal{S}_0 \) are homotopy equivalent, and thus so are the complexes \( \text{Hom}_R(\mathcal{S}, H) \) and \( \text{Hom}_R(\mathcal{S}_0, H) \) for every (Gorenstein injective) module \( H \). Hence it suffices to show the exactness of \( \text{Hom}_R(\mathcal{S}_0, H) \) whenever \( H \) is Gorenstein injective.

Now let \( H \) be any Gorenstein injective module. We need to prove the exactness of

\[
\text{Hom}_R(\mathcal{G}^+, H) \longrightarrow \text{Hom}_R(\mathcal{S}_0, H) \longrightarrow 0.
\]

To see this, let \( \alpha : K' \to H \) be any homomorphism. We wish to find \( g : G' \to H \) such that \( g \alpha = \alpha \). Now pick an exact sequence

\[
0 \longrightarrow \bar{H} \longrightarrow E \longrightarrow H \longrightarrow 0,
\]

where \( E \) is injective, and \( \bar{H} \) is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines \( H \)). Since \( \bar{H} \) is Gorenstein injective and \( \text{pd}_R K' < \infty \), we get \( \text{Ext}_R^1(K', \bar{H}) = 0 \) by [7, Lemma 1.3], and thus a lifting \( \varepsilon : K' \to E \) with \( g \varepsilon = \alpha \):

Next, injectivity of \( E \) gives \( \overline{\varepsilon} : G' \to E \) with \( \varepsilon \overline{\varepsilon} = \varepsilon \). Now \( g \overline{\varepsilon} : G' \to H \) is the desired map.

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{G}I$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G,H^+)$ is exact for all Gorenstein projective modules $G$. $\square$

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{G}I}(M,N),$$

which are functorial in $M$ and $N$. $\square$

3.7 (Definition of $\text{GExt}$). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M,N) := \text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{G}I}(M,N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare $\text{GExt}$ with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_R(M,N)$ under each of the conditions

$$\begin{align*}
& (\dagger) \text{ pd}_R M < \infty \quad \text{or} \quad (\ddagger) \text{ M } \in \text{LeftRes}_M(\mathcal{GP}) \quad \text{and} \quad \text{id}_R N < \infty. \\
& (\daggerii) \text{ There are natural isomorphisms } \text{Ext}^n_{\mathcal{G}I}(M,N) \cong \text{Ext}^n_R(M,N) \quad \text{under each of the conditions} \\
& (\ddaggeri) \text{ id}_R N < \infty \quad \text{or} \quad (\ddaggerii) \text{ N } \in \text{RightRes}_M(\mathcal{G}I) \quad \text{and} \quad \text{ pd}_R M < \infty. \\
& (\ddaggeriiii) \text{ Assume that } \text{Gpd}_R M < \infty \text{ and } \text{Gid}_R N < \infty. \text{ If either } \text{pd}_R M < \infty \text{ or } \text{id}_R N < \infty, \text{ then} \\
& \text{GExt}^n_R(M,N) \cong \text{Ext}^n_R(M,N) \\
& \text{is functorial in } M \text{ and } N.
\end{align*}$$

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{GP}$-resolution of $M$, and thus

$$\text{Ext}^n_{\mathcal{GP}}(M,N) = \text{Hom}^n(\text{Hom}_R(P,N)) = \text{Ext}^n_R(M,N).$$

In the case where $M \in \text{LeftRes}_M(\mathcal{GP})$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-,N)$, that is, $\text{Ext}^i_R(G,N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0,\ldots,Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-,N)$, we get $\text{Ext}^i_R(G,N) \cong \text{Ext}^{m+i}_R(C,N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A implies that $\text{Ext}^n_R(-,N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt". $\square$
4. Gorenstein deriving — $\otimes_R$ —

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $C$ is any of the categories in Notation 1.1 (M, $GP$, etc.), we write $rC$, respectively, $C_R$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $C$.

Now we consider the functor $\otimes_R: M_R \times R M \to A$. For fixed $M \in M_R$ and $N \in R M$ we define, in the sense of section 2.4:

\[
\text{Tor}^R_{GP}(N, M) := L^R_{GP}(N \otimes_R M) \quad \text{and} \quad \text{Tor}^R_{GP}(M, N) := L^R_{GP}(M \otimes_R N),
\]

together with

\[
\text{Tor}^R_{GP}(N, M) := L^R_{GP}(N \otimes_R M) \quad \text{and} \quad \text{Tor}^R_{GP}(M, N) := L^R_{GP}(M \otimes_R N).
\]

The first two $\text{Tor}s$ use proper left Gorenstein projective resolutions, and the last two $\text{Tor}s$ use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tor}s$, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \widetilde{X}) = (GP_R, \widetilde{GP}_R)$ or $(GP_R, \widetilde{GP}_R)$, together with

\[
(X, \widetilde{X}) = (GP_R, \widetilde{GP}_R) \quad \text{or} \quad (GP_R, \widetilde{GP}_R),
\]

and

\[
(Y, \widetilde{Y}) = (GP_R, \widetilde{GP}_R) \quad \text{or} \quad (GP_R, \widetilde{GP}_R),
\]

namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

\[
\text{LeftFPD}(R) = \sup\{pd_R M \mid M \text{ is a left } R\text{-module with } pd_R M < \infty\}.
\]

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $GP_R \subseteq GF_R$.

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_Z(M, Q/Z)$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}^R_{i}(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Theorem 4.8. Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with Gpd$_R M < \infty$, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $R$GFD-resolution of $M$ (which exists by Proposition 4.5.1). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite RightFPD$(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_2(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_2(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat. ⊓⊔

Lemma 4.7. Assume that $R$ is right coherent with finite LeftFPD$(R)$. Let $M$ be a left $R$-module with Gfd$_R M < \infty$, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $R$GFD-resolution of $M$ (which exists by Proposition 4.5.1 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite RightFPD$(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since Gfd$_R M < \infty$ and $R$ is right coherent, Proposition 4.3 gives a special short exact sequence $0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0$, where $G' = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ is an $R$GFD-resolution of $M$, and fd$_R K' < \infty$. Since $R$ has LeftFPD$(R) < \infty$, Proposition 6] implies that also pd$_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_2(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_2(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii). ⊓⊔

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
\[ \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N). \]

(ii) If $\text{Gpd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
\[ \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N) \cong \text{Tor}^r_n(M, N). \]

(iii) If $\text{Gfd}_RM < \infty$ and Gpd$_RN < \infty$, then
\[ \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N) \cong \text{Tor}^r_n(M, N). \]

(iv) If Gpd$_RM < \infty$ and Gpd$_RN < \infty$, then
\[ \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N) \cong \text{Tor}^r_n(M, N). \]

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. 

\[ \square \]

4.9 (Definition of $g\text{Tor}$ and $G\text{Tor}$). Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If Gfd$_RM < \infty$ and Gfd$_RN < \infty$, then we write
\[ g\text{Tor}^R_n(M, N) := \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(i). If Gpd$_RM < \infty$ and Gpd$_RN < \infty$, then we write
\[ G\text{Tor}^R_n(M, N) := \text{Tor}^{G\mathcal{F}}_n(M, N) \cong \text{Tor}^r_n(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. For every right $R$-module $M$ with finite Gpd$_RM$, and for every left $R$-module $N$ with Gpd$_RN < \infty$, we have isomorphisms:
\[ g\text{Tor}^R_n(M, N) \cong G\text{Tor}^R_n(M, N) \]
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $G\text{Tor}$) with the usual Tor.

**Theorem 4.11.** Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. Furthermore, let $M$ be a right $R$-module with Gfd$_RM < \infty$, and let $N$ be a left $R$-module with Gfd$_RN < \infty$. If either pd$_RM < \infty$ or pd$_RN < \infty$, then there are isomorphisms
\[ g\text{Tor}^R_n(M, N) \cong \text{Tor}^r_n(M, N) \]
that are functorial in $M$ and $N$.

Proof. If pd$_RM < \infty$, then we also have pd$_RM < \infty$ by [13 Proposition 6] (since RightFPD$(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 4.8, $P$ is also a proper left $G\mathcal{P}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
\[ g\text{Tor}^R_n(M, N) = \text{Tor}^{G\mathcal{F}}_n(M, N) = H_n(P \otimes_R N) = \text{Tor}^r_n(M, N), \]
as desired. 

\[ \square \]
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References


Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK– Danmark
E-mail address: holm@math.ku.dk