Gorenstein derived functions
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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-, -)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}_R^n(-, -)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimensional, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:
- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.
Now, given our base ring $R$, the usual right derived functors $\text{Ext}^n_R(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{\mathcal{GP}}(\cdot, \cdot)$ and $\text{Ext}^n_{\mathcal{GI}}(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{GP}$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}^n_{\mathcal{GP}}(M, N) := H^n(\text{Hom}_R(G, N)).$$

From [2.3] it will follow that $\text{Ext}^n_{\mathcal{GP}}(\cdot, \cdot)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(\mathcal{GP})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{\mathcal{GI}}(M', \cdot)$, which is defined on the full subcategory, $\text{RightRes}_M(\mathcal{GI})$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\mathcal{GP})$ and $N \in \text{RightRes}_M(\mathcal{GI})$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gfi}_R M < \infty$ for all $R$-modules $M$; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12, Proposition 2.18] (which is restated in this paper as Proposition 3.1) implies that the category $\text{LeftRes}_M(\mathcal{GP})$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\text{RightRes}_M(\mathcal{GI})$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\cdot \otimes_R \cdot$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GF}$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T: \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A proper left \( \mathcal{X} \)-resolution of \( M \in \mathcal{C} \) is a complex \( X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \rightarrow M \), such that \( X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) is also a complex, and such that the sequence

\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \rightarrow \text{Hom}_\mathcal{C}(X, X_1) \rightarrow \text{Hom}_\mathcal{C}(X, X_0) \rightarrow \text{Hom}_\mathcal{C}(X, M) \rightarrow 0
\]

is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) as an augmented proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \).

Proper right \( \mathcal{X} \)-resolutions are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_\mathcal{C}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let \( f : M \rightarrow M' \) be a morphism in \( \mathcal{C} \), and consider the diagram

\[
\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0
\]

\[
\cdots \rightarrow X'_2 \rightarrow X'_1 \rightarrow X'_0 \rightarrow M' \rightarrow 0
\]

where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n : X_n \rightarrow X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).

(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2]. \( \square \)

**Remark 2.3.** A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi : X \rightarrow M \), where \( X \in \mathcal{X} \), such that the sequence

\[
\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', M) \rightarrow 0
\]

is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1}) \), \( i > 0 \), and \( X_0 \rightarrow M \) are \( \mathcal{X} \)-precovers.

- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor \( T : \mathcal{C} \rightarrow \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n^{th} \) left derived functor

\[
L_n^\mathcal{X}T : \text{LeftRes}_\mathcal{C}(\mathcal{X}) \rightarrow \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L^n_{\mathcal{X}}T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$. Similarly, the $n^{\text{th}}$ right derived functor

$$R^n_{\mathcal{X}}T: \text{RightRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^n_{\mathcal{X}}T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_\mathcal{C}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4

$$R^n_{\mathcal{X}}F(-,N): \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^n_{\mathcal{Y}}F(M,-): \text{RightRes}_\mathcal{D}(\mathcal{Y}) \to \mathcal{E}.$$ 

If furthermore $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $N \in \text{RightRes}_\mathcal{D}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^n_{\mathcal{X}}F(M,N) \cong R^n_{\mathcal{Y}}F(M,N),$$

functorial in $M$ and $N$. Here we wrote $R^n_{\mathcal{X}}F(M,N)$ for the functor $R^n_{\mathcal{X}}F(-,N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^n_{\mathcal{X}}F(-,N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $\text{RightRes}_\mathcal{D}(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \bar{\mathcal{X}}$ and $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$.

(ii) Every $M \in \bar{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M,Y) \to F(X_0,Y) \to F(X_1,Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \bar{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X,N) \to F(X,Y^0) \to F(X,Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^n_{\mathcal{X}}F(M,N) \cong R^n_{\mathcal{Y}}F(M,N),$$

for all $M \in \bar{\mathcal{X}}$ and $N \in \bar{\mathcal{Y}}$. 


Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows
from the construction. The functoriality becomes more clear if one consults the
proof of [9, Proposition 8:2], or the proofs of [14, Theorems 2.7.2 and 2.7.6]. \square

In the next paragraphs we apply the results above to special categories
\( \mathcal{X}, \mathcal{X'}, \mathcal{C}, \mathcal{Y}, \mathcal{Y'}, \mathcal{D} \), including the categories mentioned in [11]. For completeness we
include a definition of Gorenstein projective, Gorenstein injective and Gorenstein
flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective
modules,

\[ P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots, \]

such that \( \text{Hom}_R(P, Q) \) is exact for every projective \( R \)-module \( Q \). An \( R \)-module \( M \)
is called Gorenstein projective (G-projective for short), if there exists a complete
projective resolution \( P \) with \( M \cong \text{Im}(P_0 \to P_{-1}) \). Gorenstein injective (G-injective
for short) modules are defined dually.

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for short) modules are defined dually.

In this case we define, in the sense of section 2.4,
\[ \text{Ext}_{\mathcal{GP}}^n(-, N) = \text{R}_{\mathcal{GP}}^n \text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{\mathcal{GI}}^n(M, -) = \text{R}_{\mathcal{GI}}^n \text{Hom}_R(M, -), \]

for fixed \( R \)-modules \( M \) and \( N \). We wish, of course, to apply Theorem 2.6 to this
situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If \( M \) is an \( R \)-module with \( \text{Gpd}_R M < \infty \), then there exists a
short exact sequence \( 0 \to K \to G \to M \to 0 \), where \( G \to M \) is a \( \mathcal{GP} \)-precover of
\( M \) (please see Remark [2.3]), and \( \text{pd}_R K = \text{Gpd}_R M - 1 \) (in the case where \( M \) is
Gorenstein projective, this should be interpreted as \( K = 0 \)).

Consequently, every \( R \)-module with finite Gorenstein projective dimension has a
proper left \( \mathcal{GP} \)-resolution (that is, there is an inclusion \( \mathcal{GP} \subseteq \text{LeftRes}_M(\mathcal{GP}) \)).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let \( M \) be any \( R \)-module with \( \text{Gpd}_R M < \infty \). Then
\[ \text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}_{\mathcal{GP}}^n(M, L) \neq 0 \text{ for some } R \text{-module } L \text{ with } \text{pd}_R L < \infty\}. \]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by

$$\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \}.$$ 

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences

$$S = 0 \to K \to G \to M \to 0,$$

where $G \to M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

$$S' = 0 \to K' \to G' \to M \to 0,$$

where $\pi : G' \to M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \longrightarrow 0.$$

To see this, let $\alpha : K' \to H$ be any homomorphism. We wish to find $g : G' \to H$ such that $g \iota = \alpha$. Now pick an exact sequence

$$0 \longrightarrow \tilde{H} \longrightarrow E \xrightarrow{g} H \longrightarrow 0,$$

where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $H$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon : K' \to E$ with $g \varepsilon = \alpha$:

$$\begin{array}{ccc}
K' & \xrightarrow{\alpha} & G' \\
\downarrow \varepsilon & & \downarrow \end{array}$$

Next, injectivity of $E$ gives $\tilde{\varepsilon} : G' \to E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g = g \tilde{\varepsilon} : G' \to H$ is the desired map. \hfill $\square$

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \rightarrow N \rightarrow H^0 \rightarrow H^1 \rightarrow \cdots$ be an augmented proper right $G$-resolution of $N$ (which exists by the dual of Proposition 3.1). Then Hom$_R(G, H^+)$ is exact for all Gorenstein projective modules $G$. 

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with Gpd$_R M < \infty$ and Gid$_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N),$$

which are functorial in $M$ and $N$. 

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with Gpd$_R M < \infty$ and Gid$_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms Ext$^n_{GP}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(\text{i}) \quad \text{pd}_R M < \infty \quad \text{or} \quad (\text{i}) \quad M \in \text{LeftRes}_M(GP) \quad \text{and} \quad \text{id}_R N < \infty.

(ii) There are natural isomorphisms Ext$^n_{G^2}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(\text{i}) \quad \text{id}_R N < \infty \quad \text{or} \quad (\text{i}) \quad N \in \text{RightRes}_M(GT) \quad \text{and} \quad \text{pd}_R M < \infty.

(iii) Assume that Gpd$_R M < \infty$ and Gid$_R N < \infty$. If either pd$_R M < \infty$ or id$_R N < \infty$, then

$$\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$$

is functorial in $M$ and $N$.

Proof. (i) Assume that pd$_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $GP$-resolution of $M$, and thus

$$\text{Ext}^n_{GP}(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and id$_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor Hom$_R(-, N)$, that is, Ext$^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \rightarrow G \rightarrow Q^0 \rightarrow \cdots \rightarrow Q^{m-1} \rightarrow C \rightarrow 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying Hom$_R(-, N)$, we get Ext$^i_R(G, N) \cong \text{Ext}^{m+1}_R(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A] implies that Ext$^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt$^n_R(-, -)$. 

\[\square\]
4. Gorenstein deriving $- \otimes_R -$  

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $C$ is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP},$ etc.), we write $\mathcal{R}C,$ respectively, $\mathcal{C}R,$ for the category of left, respectively, right, $R$-modules with the property describing the modules in $C.$

Now we consider the functor $\mathcal{GP}_R,$ $\mathcal{GF}_R:$  

$$\operatorname{Tor}^{\mathcal{GP}}_n(-, N) := \mathcal{L}^{\mathcal{GP}}_n(- \otimes_R N)$$  

and  

$$\operatorname{Tor}^{\mathcal{GF}}_n(M, -) := \mathcal{L}^{\mathcal{GF}}_n(M \otimes_R -),$$

together with  

$$\operatorname{Tor}^{\mathcal{GP}}_n(-, N) := \mathcal{L}^{\mathcal{GP}}_n(- \otimes_R N)$$  

and  

$$\operatorname{Tor}^{\mathcal{GF}}_n(M, -) := \mathcal{L}^{\mathcal{GF}}_n(M \otimes_R -).$$

The first two $\operatorname{Tor}$s use proper left Goreinstein projective resolutions, and the last two $\operatorname{Tor}$s use proper left Gorenstein flat resolutions. In order to compare these different $\operatorname{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \tilde{X}) = (\mathcal{GP}_R, \tilde{\mathcal{GP}}_R)$ or $(\mathcal{GF}_R, \tilde{\mathcal{GF}}_R),$ and  

and  

$(Y, \tilde{Y}) = (\mathcal{GP}_R, \tilde{\mathcal{GP}}_R)$ or $(\mathcal{GF}_R, \tilde{\mathcal{GF}}_R),$ namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\operatorname{LeftFPD}(R)$ of $R$ is defined as  

$$\operatorname{LeftFPD}(R) = \sup\{\operatorname{pd}_R M \mid M \text{ is a left } R\text{-module with } \operatorname{pd}_R M < \infty\}.$$  

The right finitistic projective dimension $\operatorname{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\operatorname{LeftFPD}(R)$ and $\operatorname{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R,$ by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\operatorname{LeftFPD}(R),$ then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathcal{GP}_R \subseteq \mathcal{GP}_R.$  

**Theorem 4.4.** For any left $R$-module $M,$ we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\operatorname{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\operatorname{Tor}_i^R (I, M) = 0$ for all injective right $R$-modules $I,$ and all $i > 0.$

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.  

$\square$
**Proposition 4.5.** Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R$-$\mathcal{G}\mathcal{F}$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R$-$\mathcal{G}\mathcal{F}$-resolution (that is, there is an inclusion $R\mathcal{G}\mathcal{F} \subseteq \text{LeftRes}_R(M(R\mathcal{G}\mathcal{F}))$).

Our first result is:

**Lemma 4.6.** Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R$-$\mathcal{P}\mathcal{D}$-resolution of $M$ (which exists by Proposition 3.1). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

**Proof.** (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.4. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat.

**Lemma 4.7.** Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R$-$\mathcal{G}\mathcal{F}$-resolution of $M$ (which exists by Proposition 4.4 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

**Proof.** (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 3.5 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R$-$\mathcal{G}\mathcal{F}$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, Proposition 6 implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$.

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 3.5 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

**Theorem 4.8.** Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[ \text{Tor}_n^{GF}(M, N) \cong \text{Tor}_n^{GF}(M, N). \]

(ii) If $\text{Gpd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[ \text{Tor}_n^{GP}(M, N) \cong \text{Tor}_n^{GF}(M, N) \cong \text{Tor}_n^{GF}(M, N). \]

(iii) If $\text{Gfd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[ \text{Tor}_n^{GF}(M, N) \cong \text{Tor}_n^{GP}(M, N) \cong \text{Tor}_n^{GF}(M, N). \]

(iv) If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[ \text{Tor}_n^{GP}(M, N) \cong \text{Tor}_n^{GF}(M, N) \cong \text{Tor}_n^{GF}(M, N). \]

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. □

4.9 (Definition of $g\text{Tor}$ and $G\text{Tor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then we write
\[ g\text{Tor}_n^R(M, N) := \text{Tor}_n^{GF}(M, N) \cong \text{Tor}_n^{GF}(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then we write
\[ G\text{Tor}_n^R(M, N) := \text{Tor}_n^{GP}(M, N) \cong \text{Tor}_n^{GP}(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_R M$, and for every left $R$-module $N$ with $\text{Gpd}_R N < \infty$, we have isomorphisms:
\[ g\text{Tor}_n^R(M, N) \cong G\text{Tor}_n^R(M, N) \]
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $G\text{Tor}$) with the usual Tor.

Theorem 4.11. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_R M < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_R N < \infty$. If either $\text{fd}_R M < \infty$ or $\text{fd}_R N < \infty$, then there are isomorphisms
\[ g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N) \]
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_R M < \infty$, then we also have $\text{pd}_R M < \infty$ by [13, Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 4.8, $P$ is also a proper left $\mathcal{GP}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
\[ g\text{Tor}_n^R(M, N) = \text{Tor}_n^{GP_R}(M, N) = H_n(P \otimes_R N) = \text{Tor}_n^R(M, N), \]
as desired. □
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References


