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GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring \( R \) it is standard to derive \( \text{Hom}_R(-,-) \) using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains \( \text{Ext}^n_R(-,-) \) in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product \( - \otimes_R - \) using Gorenstein flat modules.

1. Introduction

When \( R \) is a two-sided Noetherian ring, Auslander and Bridger \cite{AB69} introduced in 1969 the G-dimension, \( \text{G-dim}_R M \), for every finite (that is, finitely generated) \( R \)-module \( M \). They proved the inequality \( \text{G-dim}_R M \leq \text{pd}_R M \), with equality \( \text{G-dim}_R M = \text{pd}_R M \) when \( \text{pd}_R M < \infty \), along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring \( R \), Enochs and Jenda in \cite{EJ88} defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if \( R \) is two-sided Noetherian, and \( G \) is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following \cite[Theorem (4.2.6)]{AB69}. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary \( R \)-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- \( R \) is an associative ring. All modules are—if not specified otherwise—left \( R \)-modules, and the category of all \( R \)-modules is denoted \( \mathcal{M} \). We use \( \mathcal{A} \) for the category of abelian groups (that is, \( \mathbb{Z} \)-modules).
- We use \( \mathcal{GP}, \mathcal{GI} \) and \( \mathcal{GF} \) for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat \( R \)-modules; please see \cite{EJ88} and \cite{AFMR04}, or Definition 2.7 below.
- Furthermore, for each \( R \)-module \( M \) we write \( \text{Gpd}_R M, \text{Gid}_R M \) and \( \text{Gfd}_R M \) for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of \( M \), respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}_R^n(-, -)$ of $\text{Hom}_R(-, -)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{GP}(-, -)$ and $\text{Ext}^n_{GI}(-, -)$ of $\text{Hom}_R(-, -)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $GP$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see 2.1 below for the definition of proper resolutions), we define

$$\text{Ext}^n_{GP}(M, N) := \text{Ext}^n(G(M), N).$$

From 2.1 it will follow that $\text{Ext}^n_{GP}(-, N)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(GP)$, of $M$, consisting of all $R$-modules that have a proper left $GP$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{GI}(M', -)$, which is defined on the full subcategory, $\text{RightRes}_M(GI)$, of $M$, consisting of all $R$-modules that have a proper right $GI$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{GI}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(GP)$ and $N \in \text{RightRes}_M(GI)$. The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda \cite{EnochsJenda92} Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{GI}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gfd}_R M < \infty$ for all $R$-modules $M$; please see \cite{EnochsJenda92} Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, \cite{Gorenstein68} Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(GP)$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $GP$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $GI$-resolution. So $\text{RightRes}_M(GI)$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{GI}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\cdots \otimes_R \cdots$, using proper left $GP$-resolutions and proper left $GF$-resolutions. This has also been proved by Enochs and Jenda \cite{EnochsJenda92} Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $X^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence
\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \to \text{Hom}_\mathcal{C}(X, X_1) \to \text{Hom}_\mathcal{C}(X, X_0) \to \text{Hom}_\mathcal{C}(X, M) \to 0
\]
is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_\mathcal{C}(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

Proposition 2.2. Let $f: M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram
\[
\begin{array}{cccccc}
\cdots & \to & X_2 & \to & X_1 & \to & X_0 & \to & M & \to & 0 \\
& & f_2 & & f_1 & & f_0 & & f & & \\
\cdots & \to & X'_2 & \to & X'_1 & \to & X'_0 & \to & M' & \to & 0 \\
\end{array}
\]
where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \to X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

Proof. The proof is an exercise; please see [9, Exercise 8.1.2].

Remark 2.3. A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precover of $M \in \mathcal{C}$ is a morphism $\varphi: X \to M$, where $X \in \mathcal{X}$, such that the sequence
\[
\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', M) \to 0
\]
is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \to \text{Ker}(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor
\[
L_n^X T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L_n^\mathcal{X}T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$. Similarly, the $n$th right derived functor

$$R_n^\mathcal{X}T : \text{RightRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R_n^\mathcal{X}T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_\mathcal{C}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in [2.4]

$$R_n^\mathcal{X}F(-,N) : \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R_n^\mathcal{Y}F(M,-) : \text{RightRes}_\mathcal{D}(\mathcal{Y}) \to \mathcal{E}.$$  

If furthermore $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $N \in \text{RightRes}_\mathcal{D}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R_n^\mathcal{X}F(M,N) \cong R_n^\mathcal{Y}F(M,N),$$

functorial in $M$ and $N$. Here we wrote $R_n^\mathcal{X}F(M,N)$ for the functor $R_n^\mathcal{X}F(-,N)$ applied to $M$. Another, and perhaps better, notation could be

$$R_n^\mathcal{X}(\mathcal{X},-,N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $\text{RightRes}_\mathcal{D}(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \bar{\mathcal{X}}$ and $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$.
(ii) Every $M \in \bar{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M,Y) \to F(X_0,Y) \to F(X_1,Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.
(iii) Every $N \in \bar{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X,N) \to F(X,Y^0) \to F(X,Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R_n^\mathcal{X}F(M,N) \cong R_n^\mathcal{Y}F(M,N),$$

for all $M \in \bar{\mathcal{X}}$ and $N \in \bar{\mathcal{Y}}$. 
Proof. Please see \cite[Proposition 2.3]{T}. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of \cite[Proposition 8.2.14]{T}, or the proofs of \cite[Theorems 2.7.2 and 2.7.6]{T}. □

In the next paragraphs we apply the results above to special categories $X$, $\tilde{X}$, $\mathcal{C}$ and $\tilde{Y}$, $\mathcal{D}$, including the categories mentioned in \ref{subsection}. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

$$P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \to P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\mathcal{G}P$, $\mathcal{G}I$ and $\mathcal{G}F$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $G$-precovers are always surjective, and $\mathcal{G}P$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -): \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} = \mathcal{G}P, \quad \tilde{X} = \mathcal{G}\tilde{P} \quad \text{and} \quad \mathcal{Y} = \mathcal{G}I, \quad \tilde{Y} = \mathcal{G}\tilde{I}.$$

In this case we define, in the sense of section \ref{subsection},

$$\text{Ext}_{\mathcal{G}P}^n(-, N) = R_{\mathcal{G}P}^n\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{\mathcal{G}I}^n(M, -) = R_{\mathcal{G}I}^n\text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem \ref{theorem} to this situation. Note that by \cite[Proposition 2.18]{T}, we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_RM < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $\mathcal{G}P$-precover of $M$ (please see Remark \ref{remark}), and $\text{pd}_R K = \text{Gpd}_RM - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $\mathcal{G}P$-resolution (that is, there is an inclusion $\mathcal{G}P \subseteq \text{LeftRes}_M(\mathcal{G}P)$).

Furthermore, we will need the following from \cite[Theorem 2.13]{T}:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_RM < \infty$. Then

$$\text{Gpd}_RM = \sup\{n \geq 0 \mid \text{Ext}_{\mathcal{G}P}^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_RL < \infty\}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by
\[ \text{pd}_R M = \{ n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \}. \]
It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $GP$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $GP$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences
\[ S = 0 \to K \to G \to M \to 0, \]
where $G \to M$ is a $GP$-precover of some module $M$ with Gpd$R M < \infty$ (recall that $GP$-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,
\[ S = 0 \to K \to G \to M \to 0, \]
where $\pi$: $G \to M$ is a $GP$-precover and $\text{pd}_R K < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of
\[ \text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \to 0. \]
To see this, let $\alpha$: $K' \to H$ be any homomorphism. We wish to find $g$: $G' \to H$ such that $g \iota = \alpha$. Now pick an exact sequence
\[ 0 \to \tilde{H} \to E \xrightarrow{g} H \to 0, \]
where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K < \infty$, we get $\text{Ext}_R^n(K', \tilde{H}) = 0$ by Lemma 1.3, and thus a lifting $\varepsilon$: $K' \to E$ with $g \varepsilon = \alpha$:

Next, injectivity of $E$ gives $\tilde{\varepsilon}$: $G' \to E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g \tilde{\varepsilon}$: $G' \to H$ is the desired map. \qed

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{G}I$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G,H^+) = 0$ is exact for all Gorenstein projective modules $G$. □

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{G}I}(M,N),$

which are functorial in $M$ and $N$. □

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$\text{GExt}^n_R(M,N) := \text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{G}I}(M,N)$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_R(M,N)$ under each of the conditions

$(a) \text{pd}_R M < \infty$ or $(b) M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N < \infty$.

(ii) There are natural isomorphisms $\text{Ext}^n_{\mathcal{G}I}(M,N) \cong \text{Ext}^n_R(M,N)$ under each of the conditions

$(a) \text{id}_R N < \infty$ or $(b) N \in \text{RightRes}_M(\mathcal{G}I)$ and $\text{pd}_R M < \infty$.

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then

$\text{GExt}^n_R(M,N) = \text{Ext}^n_R(M,N)$

is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{GP}$-resolution of $M$, and thus

$\text{Ext}^n_{\mathcal{GP}}(M,N) = \text{Ext}^n_R(M,N) = 0$(1)

In the case where $M \in \text{LeftRes}_M(\mathcal{GP})$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-,N)$, that is, $\text{Ext}^i_R(G,N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-,N)$, we get $\text{Ext}^i_R(G,N) \cong \text{Ext}^{m+i}_R(C,N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A implies that $\text{Ext}^n_R(-,N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $\text{GExt}^n_R(-,-)$. □
4. Gorenstein deriving $- \otimes_R -$  

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed: 

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP}, \text{etc.}$), we write $r\mathcal{C}$, respectively, $\mathcal{C}r$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $- \otimes_R -$ : $\mathcal{M}_R \times R\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in R\mathcal{M}$ we define, in the sense of section 2.4: 

$$\text{Tor}^{\mathcal{GP}}_n(-, N) := \text{L}^{\mathcal{GP}}_n(- \otimes_R N) \quad \text{and} \quad \text{Tor}^{\mathcal{GP}}_n(M, -) := \text{L}^{\mathcal{GP}}_n(M \otimes_R -),$$

and

$$\text{Tor}^{\mathcal{GF}}_n(-, N) := \text{L}^{\mathcal{GF}}_n(- \otimes_R N) \quad \text{and} \quad \text{Tor}^{\mathcal{GF}}_n(M, -) := \text{L}^{\mathcal{GF}}_n(M \otimes_R -).$$

The first two $\text{Tor}$s use proper left Gorenstein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein at resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \mathcal{X}) = (\mathcal{GP}_R, \mathcal{GP}_R)$ or $(\mathcal{GF}_R, \mathcal{GF}_R)$, respectively, the covariant-covariant version of Theorem 2.6, instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$$\text{LeftFPD}(R) = \sup\{\text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty\}.$$ 

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathcal{GP} \subseteq \mathcal{GF}$. 

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}^R_i(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}F$-resolution (that is, there is an inclusion $R\mathcal{G}F \subseteq \text{LeftRes}_{R,R}(R\mathcal{G}F)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}P$-resolution of $M$ (which exists by Proposition 3.4). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_Z(T, Q/Z)$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, Q/Z)$ is exact by Proposition 4.5. Since $Q/Z$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat. 

Lemma 4.7. Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition 4.5 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 4.8 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, 4.5. Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, Q/Z)$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of 

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, Q/Z).$$

Since $Q/Z$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
$$\text{Tor}_n^{G\mathcal{P}}(M, N) \cong \text{Tor}_n^{G\mathcal{F}}(M, N).$$

(ii) If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
$$\text{Tor}_n^{G\mathcal{P}}(M, N) \cong \text{Tor}_n^{G\mathcal{F}}(M, N) \cong \text{Tor}_n^{G\mathcal{F} R}(M, N).$$

(iii) If $\text{Gfd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
$$\text{Tor}_n^{G\mathcal{P}}(M, N) \cong \text{Tor}_n^{G\mathcal{F}}(M, N) \cong \text{Tor}_n^{G\mathcal{F} R}(M, N).$$

(iv) If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
$$\text{Tor}_n^{G\mathcal{P}}(M, N) \cong \text{Tor}_n^{G\mathcal{F}}(M, N) \cong \text{Tor}_n^{G\mathcal{F} R}(M, N).$$

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6.

4.9 (Definition of $g\text{Tor}$ and $G\text{Tor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then we write
$$g\text{Tor}_n^R(M, N) := \text{Tor}_n^{G\mathcal{F}}(M, N) \cong \text{Tor}_n^{G\mathcal{F} R}(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then we write
$$G\text{Tor}_n^R(M, N) := \text{Tor}_n^{G\mathcal{P}}(M, N) \cong \text{Tor}_n^{G\mathcal{P} R}(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_RM$, and for every left $R$-module $N$ with $\text{Gpd}_RN < \infty$, we have isomorphisms:
$$g\text{Tor}_n^R(M, N) \cong G\text{Tor}_n^R(M, N)$$
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $G\text{Tor}$) with the usual Tor.

Theorem 4.11. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_RM < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_RN < \infty$. If either $\text{fd}_RM < \infty$ or $\text{fd}_RN < \infty$, then there are isomorphisms
$$g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)$$
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_RM < \infty$, then we also have $\text{pd}_RM < \infty$ by [13, Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 3.3 $P$ is also a proper left $G\mathcal{P} R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
$$g\text{Tor}_n^R(M, N) = \text{Tor}_n^{G\mathcal{P}}(M, N) = H_n(P \otimes_R N) = \text{Tor}_n^R(M, N),$$
as desired.\[\square\]
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References


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