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GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-, -)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}^n_R(-, -)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring $R$, the usual right derived functors $\Ext^n_R(-,-)$ of $\Hom_R(-,-)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\Ext^n_{GP}(-,-)$ and $\Ext^n_{GI}(-,-)$ of $\Hom_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $GP$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\Ext^n_{GP}(M, N) := H^n(\Hom_R(G, N)).$$

From [2.3] it will follow that $\Ext^n_{GP}(-,-)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(GP)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $GP$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\Ext^n_{GP}(M', -)$, which is defined on the full subcategory, $\text{RightRes}_M(GI)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper right $GI$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\Ext^n_{GP}(M, N) \cong \Ext^n_{GI}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(GP)$ and $N \in \text{RightRes}_M(GI)$. The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\Ext^n_{GP}(M, N) \cong \Ext^n_{GI}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M \leq n$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12, Proposition 2.18] (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(GP)$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $GP$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $GI$-resolution. So $\text{RightRes}_M(GI)$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\Ext^n_{GP}(M, N) \cong \Ext^n_{GI}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $GP$-resolutions and proper left $GF$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A proper left \( \mathcal{X} \)-resolution of \( M \in \mathcal{C} \) is a complex \( X = \cdots \to X_1 \to X_0 \to 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \to M \), such that \( X^+ := \cdots \to X_1 \to X_0 \to M \to 0 \) is also a complex, and such that the sequence
\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \to \text{Hom}_\mathcal{C}(X, X_1) \to \text{Hom}_\mathcal{C}(X, X_0) \to \text{Hom}_\mathcal{C}(X, M) \to 0
\]
is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \to X_1 \to X_0 \to M \to 0 \) as an augmented proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \).

Proper right \( \mathcal{X} \)-resolutions are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_\mathcal{C}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

Proposition 2.2. Let \( f: M \to M' \) be a morphism in \( \mathcal{C} \), and consider the diagram
\[
\begin{array}{ccccccccc}
\cdots & \to & X_2 & \to & X_1 & \to & X_0 & \to & M & \to & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\cdots & \to & X'_2 & \to & X'_1 & \to & X'_0 & \to & M' & \to & 0
\end{array}
\]
where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n: X_n \to X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).

(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

Proof. The proof is an exercise; please see [9, Exercise 8.1.2].

Remark 2.3. A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.
- Recall (see [15] Definition 1.2.2)) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi: X \to M \), where \( X \in \mathcal{X} \), such that the sequence
\[
\text{Hom}_\mathcal{C}(X', X) \to \text{Hom}_\mathcal{C}(X', \varphi) \to \text{Hom}_\mathcal{C}(X', M) \to 0
\]
is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \to \text{Ker}(X_i \to X_{i-1}) \), \( i > 0 \), and \( X_0 \to M \) are \( \mathcal{X} \)-precovers.
- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9] Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology proper from [3] Section 4.

2.4 (Derived Functors). Consider an additive functor \( T: \mathcal{C} \to \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n \)th left derived functor
\[
L_n^X T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}
\]
of \( T \), with respect to the class \( \mathcal{X} \), by setting \( L^n_\mathcal{X} T(M) = H_n(T(X)) \), where \( X \) is any proper left \( \mathcal{X} \)-resolution of \( M \in \text{LeftRes}_C(\mathcal{X}) \). Similarly, the \( n \)th right derived functor

\[
R^n_\mathcal{X} T : \text{RightRes}_C(\mathcal{X}) \rightarrow \mathcal{E}
\]

of \( T \) with respect to \( \mathcal{X} \) is defined by \( R^n_\mathcal{X} T(N) = H_n(T(Y)) \), where \( Y \) is any proper right \( \mathcal{X} \)-resolution of \( N \in \text{RightRes}_C(\mathcal{X}) \). These constructions are well-defined and functorial in the arguments \( M \) and \( N \) by Proposition 2.2.

The situation where \( T \) is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category \( \mathcal{D} \), together with a full subcategory \( \mathcal{Y} \subseteq \mathcal{D} \) and an additive functor \( F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \) in two variables. We will assume that \( F \) is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of \( F \) is not important, and the definitions and results below can easily be modified to fit the situation where \( F \) is covariant in both variables, say.

For fixed \( M \in \mathcal{C} \) and \( N \in \mathcal{D} \) we can then consider the two right derived functors as in 2.4:

\[
R^n_\mathcal{X} F(-, N) : \text{LeftRes}_C(\mathcal{X}) \rightarrow \mathcal{E} \quad \text{and} \quad R^n_\mathcal{Y} F(M, -) : \text{RightRes}_D(\mathcal{Y}) \rightarrow \mathcal{E}.
\]

If furthermore \( M \in \text{LeftRes}_C(\mathcal{X}) \) and \( N \in \text{RightRes}_D(\mathcal{Y}) \), we can ask for a sufficient condition to ensure that

\[
R^n_\mathcal{X} F(M, N) \cong R^n_\mathcal{Y} F(M, N),
\]

functorial in \( M \) and \( N \). Here we wrote \( R^n_\mathcal{X} F(M, N) \) for the functor \( R^n_\mathcal{X} F(-, N) \) applied to \( M \). Another, and perhaps better, notation could be

\[
R^n_\mathcal{X} F(-, N)[M].
\]

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor \( F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \) which is contravariant in the first variable and covariant in the second variable, together with the full subcategories \( \mathcal{X} \subseteq \mathcal{C} \) and \( \mathcal{Y} \subseteq \mathcal{D} \). Assume that we have full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \text{LeftRes}_C(\mathcal{X}) \) and \( \text{RightRes}_D(\mathcal{Y}) \), respectively, satisfying:

(i) \( \mathcal{X} \subseteq \bar{\mathcal{X}} \) and \( \mathcal{Y} \subseteq \bar{\mathcal{Y}} \).

(ii) Every \( M \in \bar{\mathcal{X}} \) has an augmented proper left \( \mathcal{X} \)-resolution \( \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \), such that \( 0 \rightarrow F(M, Y) \rightarrow F(X_0, Y) \rightarrow F(X_1, Y) \rightarrow \cdots \) is exact for all \( Y \in \mathcal{Y} \).

(iii) Every \( N \in \bar{\mathcal{Y}} \) has an augmented proper right \( \mathcal{Y} \)-resolution \( 0 \rightarrow N \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \), such that \( 0 \rightarrow F(X, N) \rightarrow F(X, Y^0) \rightarrow F(X, Y^1) \rightarrow \cdots \) is exact for all \( X \in \mathcal{X} \).

Then we have functorial isomorphisms

\[
R^n_\mathcal{X} F(M, N) \cong R^n_\mathcal{Y} F(M, N),
\]

for all \( M \in \bar{\mathcal{X}} \) and \( N \in \bar{\mathcal{Y}} \).
Proof. Please see \cite[Proposition 2.3]{Gorenstein Derived Functors}. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of \cite[Proposition 8.2.14]{Gorenstein Derived Functors}, or the proofs of \cite[Theorems 2.7.2 and 2.7.6]{Gorenstein Derived Functors}.

In the next paragraphs we apply the results above to special categories $X$, $\tilde{X}$, $C$ and $Y$, $\tilde{Y}$, $D$, including the categories mentioned in \ref{cat:1}. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

$$P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \to P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\widetilde{G\mathcal{P}}$, $\widetilde{G\mathcal{I}}$ and $\widetilde{G\mathcal{F}}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $G\mathcal{P}$-precovers are always surjective, and $\widetilde{G\mathcal{P}}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -): \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} = G\mathcal{P}, \quad \tilde{\mathcal{X}} = \widetilde{G\mathcal{P}}, \quad \mathcal{Y} = G\mathcal{I}, \quad \tilde{\mathcal{Y}} = \widetilde{G\mathcal{I}}.$$

In this case we define, in the sense of section \ref{sec:2.1},

$$\text{Ext}^n_{\widetilde{G\mathcal{P}}}(\cdot, -) = R^n_{\widetilde{G\mathcal{P}}} \text{Hom}_R(\cdot, -) \quad \text{and} \quad \text{Ext}^n_{\widetilde{G\mathcal{I}}}(M, -) = R^n_{\widetilde{G\mathcal{I}}} \text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by \cite[Proposition 2.18]{Gorenstein Derived Functors}, we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_RM < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $G\mathcal{P}$-precover of $M$ (please see Remark \ref{rem:2.3}), and $\text{pd}_RK = \text{Gpd}_RM - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $G\mathcal{P}$-resolution (that is, there is an inclusion $G\mathcal{P} \subseteq \text{LeftRes}_M(G\mathcal{P})$).

Furthermore, we will need the following from \cite[Theorem 2.13]{Gorenstein Derived Functors}:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_RM < \infty$. Then

$$\text{Gpd}_RM = \sup\{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_RL < \infty\}. $$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an \( R \)-module \( M \) is given by

\[
\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \}.
\]

It also follows that if \( \text{pd}_R M < \infty \), then every projective resolution of \( M \) is actually a proper left \( G_P \)-resolution of \( M \).

Lemma 3.4. Assume that \( M \) is an \( R \)-module with finite Gorenstein projective dimension, and let \( G^+ = \cdots \to G_1 \to G_0 \to M \to 0 \) be an augmented proper left \( G_P \)-resolution of \( M \) (which exists by Proposition 3.1). Then \( \text{Hom}_R(G^+, H) \) is exact for all Gorenstein injective modules \( H \).

Proof. We split the proper resolution \( G^+ \) into short exact sequences. Hence it suffices to show exactness of \( \text{Hom}_R(S, H) \) for all Gorenstein injective modules \( H \) and all short exact sequences

\[
S = 0 \to K \to G \to M \to 0,
\]

where \( G \to M \) is a \( G_P \)-precover of some module \( M \) with \( \text{Gpd}_R M < \infty \) (recall that \( G_P \)-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

\[
S' = 0 \to K' \to G' \to M \to 0,
\]

where \( \pi: G' \to M \) is a \( G_P \)-precover and \( \text{pd}_R K' < \infty \).

It is easy to see (as in Proposition 2.2) that the complexes \( S \) and \( S' \) are homotopy equivalent, and thus so are the complexes \( \text{Hom}_R(S, H) \) and \( \text{Hom}_R(S', H) \) for every (Gorenstein injective) module \( H \). Hence it suffices to show the exactness of \( \text{Hom}_R(S', H) \) whenever \( H \) is Gorenstein injective.

Now let \( H \) be any Gorenstein injective module. We need to prove the exactness of

\[
\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \xrightarrow{} 0.
\]

To see this, let \( \alpha: K' \to H \) be any homomorphism. We wish to find \( g: G' \to H \) such that \( g\iota = \alpha \). Now pick an exact sequence

\[
0 \xrightarrow{} \bar{H} \xrightarrow{g} E \xrightarrow{\theta} H \xrightarrow{} 0,
\]

where \( E \) is injective, and \( \bar{H} \) is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines \( H \)). Since \( \bar{H} \) is Gorenstein injective and \( \text{pd}_R K' < \infty \), we get \( \text{Ext}_R^1(K', \bar{H}) = 0 \) by Lemma 1.3, and thus a lifting \( \varepsilon: K' \to E \) with \( g\varepsilon = \alpha \):}

Next, injectivity of \( E \) gives \( \bar{\varepsilon}: G' \to E \) with \( \bar{\varepsilon}\iota = \varepsilon \). Now \( g = g\bar{\varepsilon}: G' \to H \) is the desired map. \( \square \)

With a similar proof we get:
Lemma 3.5. Assume that \( N \) is an \( R \)-module with finite Gorenstein injective dimension, and let \( H^+ = 0 \to N \to H^0 \to H^1 \to \cdots \) be an augmented proper right \( \mathcal{G} \)-resolution of \( N \) (which exists by the dual of Proposition 3.4). Then \( \text{Hom}_R(G, H^+) \) is exact for all Gorenstein projective modules \( G \).

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

**Theorem 3.6.** For all \( R \)-modules \( M \) and \( N \) with \( \text{Gpd}_R M < \infty \) and \( \text{Gid}_R N < \infty \), we have isomorphisms

\[
\text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}T}(M, N),
\]

which are functorial in \( M \) and \( N \).

3.7 (Definition of \( \text{GExt} \)). Let \( M \) and \( N \) be \( R \)-modules with \( \text{Gpd}_R M < \infty \) and \( \text{Gid}_R N < \infty \). Then we write

\[
\text{GExt}^n_R(M, N) := \text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}T}(M, N)
\]

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare \( \text{GExt} \) with the classical \( \text{Ext} \). This is done in:

**Theorem 3.8.** Let \( M \) and \( N \) be any \( R \)-modules. Then the following conclusions hold:

(i) There are natural isomorphisms \( \text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_R(M, N) \) under each of the conditions

\[
\begin{align*}
\text{(i)} & \quad \text{pd}_R M < \infty \quad \text{or} \quad (\dagger) \quad M \in \text{LeftRes}_M(\mathcal{G}P) \quad \text{and} \quad \text{id}_R N < \infty, \\
\text{(ii)} & \quad \text{There are natural isomorphisms} \quad \text{Ext}^n_{\mathcal{G}T}(M, N) \cong \text{Ext}^n_R(M, N) \quad \text{under each of the conditions}
\end{align*}
\]

\[
\begin{align*}
\text{(i)} & \quad \text{id}_R N < \infty \quad \text{or} \quad (\ddagger) \quad N \in \text{RightRes}_M(\mathcal{G}T) \quad \text{and} \quad \text{pd}_R M < \infty, \\
\text{(ii)} & \quad \text{Assume that} \quad \text{Gpd}_R M < \infty \quad \text{and} \quad \text{Gid}_R N < \infty. \quad \text{If either} \ \text{pd}_R M < \infty \quad \text{or} \quad \text{id}_R N < \infty, \ \text{then}
\end{align*}
\]

\[
\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)
\]

is functorial in \( M \) and \( N \).

**Proof.** (i) Assume that \( \text{pd}_R M < \infty \), and pick any projective resolution \( P \) of \( M \). By Remark 3.3, \( P \) is also a proper left \( \mathcal{G}P \)-resolution of \( M \), and thus

\[
\text{Ext}^n_{\mathcal{G}P}(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).
\]

In the case where \( M \in \text{LeftRes}_M(\mathcal{G}P) \) and \( \text{id}_R N = m < \infty \), we see that Gorenstein projective modules are acyclic for the functor \( \text{Hom}_R(-, N) \), that is, \( \text{Ext}^i_R(G, N) = 0 \) (the usual Ext) for every Gorenstein projective module \( G \), and every integer \( i > 0 \).

This is because, if \( G \) is a Gorenstein projective module, and \( i > 0 \) is an integer, then there exists an exact sequence \( 0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0 \), where \( Q^0, \ldots, Q^{m-1} \) are projective modules. Breaking this exact sequence into short exact ones, and applying \( \text{Hom}_R(-, N) \), we get \( \text{Ext}^i_R(G, N) = 0 \), as claimed.

Therefore [11], Chapter III, Proposition 1.2A implies that \( \text{Ext}^n_R(-, N) \) can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of \( \text{GExt}^n_R(-, -) \).
4. Gorenstein deriving $- \otimes_R -$ 

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $C$ is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP}$, etc.), we write $\mathfrak{r}C$, respectively, $C_R$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $C$.

Now we consider the functor $- \otimes_R -$: $\mathcal{M} \times \mathfrak{r} \mathcal{M} \rightarrow \mathcal{A}$. For fixed $M \in \mathcal{M}$ and $N \in \mathfrak{r} \mathcal{M}$ we define, in the sense of section 2.4:

\begin{align*}
\text{Tor}_n^{\mathcal{GP}}(-, N) &:= L_n^{\mathcal{GP}}(- \otimes_R N) \quad \text{and} \quad \text{Tor}_n^{\mathcal{GP}}(M, -) := L_n^{\mathcal{GP}}(M \otimes_R -), \\
\text{Tor}_n^{\mathcal{GF}}(-, N) &:= L_n^{\mathcal{GF}}(- \otimes_R N) \quad \text{and} \quad \text{Tor}_n^{\mathcal{GF}}(M, -) := L_n^{\mathcal{GF}}(M \otimes_R -).
\end{align*}

The first two $\text{Tors}$ use proper left Gorestein projective resolutions, and the last two $\text{Tors}$ use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tors}$, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $\mathfrak{r}(X, e_X) = (\mathcal{GP}, f_{\mathcal{GP}})$ or $(\mathcal{GF}, g_{\mathcal{GF}})$ and $\mathfrak{r}(Y, e_Y) = (\mathfrak{r}\mathcal{GP}, f_{\mathfrak{r}\mathcal{GP}})$ or $(\mathfrak{r}\mathcal{GF}, g_{\mathfrak{r}\mathcal{GF}})$.

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$$\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.$$ 

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathfrak{r}\mathcal{GP} \subseteq \mathfrak{r}\mathcal{GF}$. \hfill $\square$

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ consisting of flat left $R$-modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent. \hfill $\square$
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with \( \text{Gfd}_R M < \infty \), then there exists a short exact sequence \( 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0 \), where $G \rightarrow M$ is an $R\mathcal{G}$-precover of $M$, and \( \text{fd}_R K = \text{Gfd}_R M - 1 \) (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}$-resolution (that is, there is an inclusion $R\mathcal{G} \subseteq \text{LeftRes}_M(R\mathcal{G})$).

\[
\text{Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.}
\]

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with Gpd$_R M < \infty$, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $R\mathcal{GP}$-resolution of $M$ (which exists by Proposition 4.6). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite RightFPD$(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

\[
\text{Proof. (i) By Proposition 4.6 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.}
\]

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite LeftFPD$(R)$. Let $M$ be a left $R$-module with Gpd$_R M < \infty$, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $R\mathcal{GF}$-resolution of $M$ (which exists by Proposition 4.7. Since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite RightFPD$(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

\[
\text{Proof. (i) Since Gpd$_R M < \infty$ and $R$ is right coherent, Proposition 4.8 gives a special short exact sequence $0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0$, where $G' \rightarrow M$ is an $R\mathcal{GF}$-precover of $M$, and \( \text{fd}_R K' < \infty \). Since $R$ has LeftFPD$(R) < \infty$, Proposition 6] implies that also pd$_R K' < \infty$. Now the proof of Lemma 4.4 applies.}
\]

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

\[
\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).
\]

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.8 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both LeftFPD$(R)$ and RightFPD$(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
$$\text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N).$$

(ii) If $\text{Gpd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
$$\text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N) \cong \text{Tor}^F_n(M, N).$$

(iii) If $\text{Gfd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
$$\text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N) \cong \text{Tor}^F_n(M, N).$$

(iv) If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
$$\text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N) \cong \text{Tor}^F_n(M, N).$$

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6.

4.9 (Definition of $g\text{Tor}$ and $G\text{Tor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then we write
$$g\text{Tor}^R_n(M, N) := \text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then we write
$$G\text{Tor}^R_n(M, N) := \text{Tor}^G_n(M, N) \cong \text{Tor}^F_n(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_RM$, and for every left $R$-module $N$ with $\text{Gpd}_RN < \infty$, we have isomorphisms:
$$g\text{Tor}^R_n(M, N) \cong G\text{Tor}^R_n(M, N)$$
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $G\text{Tor}$) with the usual Tor.

Theorem 4.11. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_RM < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_RN < \infty$. If either $\text{fd}_RM < \infty$ or $\text{fd}_RN < \infty$, then there are isomorphisms
$$g\text{Tor}^R_n(M, N) \cong \text{Tor}^R_n(M, N)$$
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_RM < \infty$, then we also have $\text{pd}_RM < \infty$ by [13, Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 4.8, $P$ is also a proper left $\mathcal{GP}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
$$g\text{Tor}^R_n(M, N) = \text{Tor}^G_n(M, N) = H_n(P \otimes_R N) = \text{Tor}^R_n(M, N),$$
as desired.
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References


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