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GORENSTEIN DERIVED FUNCTORS

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ABSTRACT. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-,-)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}_R^n(-,-)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger \cite{2} introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in \cite{6} defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following \cite[Theorem (4.2.6)]{4}. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\text{GP}$, $\text{GI}$ and $\text{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see \cite{6} and \cite{8}, or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}_R^n(-,-)$ of $\text{Hom}_R(-,-)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}_R^n(-,-)$ and $\text{Ext}_R^n(-,-)$ of $\text{Hom}_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{GP}$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}_R^n(M,N) := H^n(\text{Hom}_R(G,N)).$$

From [2.4] it will follow that $\text{Ext}_R^n(-,-)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(\mathcal{GP})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}_R^n(M',-)$, which is defined on the full subcategory, $\text{RightRes}_M(\mathcal{GI})$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}_R^n(M,N) \cong \text{Ext}_R^n(M,N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\mathcal{GP})$ and $N \in \text{RightRes}_M(\mathcal{GI})$. The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}_R^n(M,N) \cong \text{Ext}_R^n(M,N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_RM < \infty$, $\text{Gid}_RM < \infty$, and also $\text{Gpd}_RM < \infty$ for all $R$-modules $M$; please see [9] Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12] Proposition 2.18 (which is restated in this paper as Proposition 2.1.1) implies that the category $\text{LeftRes}_M(\mathcal{GP})$ contains all $R$-modules $M$ with $\text{Gpd}_RM < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\text{RightRes}_M(\mathcal{GI})$ contains all $R$-modules $N$ with $\text{Gid}_RN < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\text{Ext}_R^n(M,N) \cong \text{Ext}_R^n(M,N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $-\otimes_R-$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GF}$-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T: \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $X \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A proper left \( \mathcal{X} \)-resolution of \( M \in \mathcal{C} \) is a complex \( X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \rightarrow M \), such that \( X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) is also a complex, and such that the sequence
\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \rightarrow \text{Hom}_\mathcal{C}(X, X_1) \rightarrow \text{Hom}_\mathcal{C}(X, X_0) \rightarrow \text{Hom}_\mathcal{C}(X, M) \rightarrow 0
\]
is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) as an augmented proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \).

Proper right \( \mathcal{X} \)-resolutions are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_\mathcal{C}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let \( f: M \rightarrow M' \) be a morphism in \( \mathcal{C} \), and consider the diagram
\[
\begin{array}{ccccccccc}
\cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\cdots & \rightarrow & X'_2 & \rightarrow & X'_1 & \rightarrow & X'_0 & \rightarrow & M' & \rightarrow & 0
\end{array}
\]
where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n: X_n \rightarrow X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).
(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2]. \( \square \)

**Remark 2.3.** A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.
- Recall (see [15] Definition 1.2.2)) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi: X \rightarrow M \), where \( X \in \mathcal{X} \), such that the sequence
\[
\text{Hom}_\mathcal{C}(X', X) \rightarrow \text{Hom}_\mathcal{C}(X', M) \rightarrow 0
\]
is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1}) \), \( i > 0 \), and \( X_0 \rightarrow M \) are \( \mathcal{X} \)-precovers.
- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9] Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology *proper* from [3] Section 4).

2.4 (Derived Functors). Consider an additive functor \( T: \mathcal{C} \rightarrow \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n \)-th left derived functor
\[
L_n^X T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \rightarrow \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L^X_n(T(M)) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_C(\mathcal{X})$. Similarly, the $n$th right derived functor

$$R^X_nT: \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^X_nT(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_C(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4

$$R^X_nF(\cdot, N): \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^Y_nF(M, \cdot): \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.$$If furthermore $M \in \text{LeftRes}_C(\mathcal{X})$ and $N \in \text{RightRes}_D(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^X_nF(M, N) \cong R^Y_nF(M, N),$$

functorial in $M$ and $N$. Here we wrote $R^X_nF(M, N)$ for the functor $R^X_nF(\cdot, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^X_nF(\cdot, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

Theorem 2.6. Consider the functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_C(\mathcal{X})$ and $\text{RightRes}_D(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \mathcal{X}^0$ and $\mathcal{Y} \subseteq \mathcal{Y}^0$.

(ii) Every $M \in \mathcal{X}^0$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \mathcal{Y}^0$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^X_nF(M, N) \cong R^Y_nF(M, N),$$

for all $M \in \mathcal{X}^0$ and $N \in \mathcal{Y}^0$. 
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories $X$, $\bar{X}$, $C$ and $\bar{Y}, \bar{D}$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

$$P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \to P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\widehat{GP}$, $\widehat{GI}$ and $\widehat{GF}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $GP$-precovers are always surjective, and $\widehat{GP}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} = \mathcal{GP}, \quad \bar{X} = \widehat{GP} \quad \text{and} \quad \mathcal{Y} = \mathcal{GI}, \quad \bar{Y} = \widehat{GI}.$$

In this case we define, in the sense of section 2.4

$$\text{Ext}^n_{\mathcal{GP}}(-, N) = R^n_{\mathcal{GP}}\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}^n_{\mathcal{GI}}(M, -) = R^n_{\mathcal{GI}}\text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with Gpd$_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $GP$-precover of $M$ (please see Remark 2.22), and pd$_R K = \text{Gpd}_R M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $GP$-resolution (that is, there is an inclusion $\widehat{GP} \subseteq \text{LeftRes}_M(GP)$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with Gpd$_R M < \infty$. Then

$$\text{Gpd}_R M = \sup \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with pd}_R L < \infty \}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by

$$\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R (M, L) \neq 0 \text{ for some } R\text{-module } L \}.$$ 

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R (G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R (S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences

$$S = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0,$$

where $G \rightarrow M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence

$$S' = 0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0,$$

where $\pi: G' \rightarrow M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R (S, H)$ and $\text{Hom}_R (S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R (S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R (G', H) \xrightarrow{\text{Hom}_R (\iota, H)} \text{Hom}_R (K', H) \rightarrow 0.$$

To see this, let $\alpha: K' \rightarrow H$ be any homomorphism. We wish to find $g: G' \rightarrow H$ such that $g \iota = \alpha$. Now pick an exact sequence

$$0 \rightarrow \tilde{H} \rightarrow E \xrightarrow{g} H \rightarrow 0,$$

where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R (K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \rightarrow E$ with $g \varepsilon = \alpha$:

![Diagram](https://via.placeholder.com/150)

Next, injectivity of $E$ gives $\tilde{\varepsilon}: G' \rightarrow E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g \tilde{\varepsilon} = G' \rightarrow H$ is the desired map. □

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{G}I$-resolution of $N$ (which exists by the dual of Proposition 3.3). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$.

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N),$$

which are functorial in $M$ and $N$.

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{itemize}
  \item [(\dag)] $\text{pd}_R M < \infty$ \quad or \quad (\dag) $M \in \text{LeftRes}_M(\mathcal{G}P)$ and $\text{id}_R N < \infty$.
\end{itemize}

(ii) There are natural isomorphisms $\text{Ext}^n_{G^2}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{itemize}
  \item [(\dag)] $\text{id}_R N < \infty$ \quad or \quad (\dag) $N \in \text{RightRes}_M(\mathcal{G}I)$ and $\text{pd}_R M < \infty$.
\end{itemize}

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then

$$\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$$

is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{G}P$-resolution of $M$, and thus

$$\text{Ext}^n_{GP}(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(\mathcal{G}P)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}_{R}^{m+i}(C, N) = 0$, as claimed.

Therefore [11 Chapter III, Proposition 1.2A] implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt$^n_R(-, -)$.
4. Gorenstein deriving \(-\otimes_R-\)

In dealing with the tensor product we need, of course, both left and right \(R\)-modules. Thus the following addition to Notation 1.1 is needed:

If \(C\) is any of the categories in Notation 1.1 (\(\mathcal{M}, \mathcal{GP}\), etc.), we write \(rC\), respectively, \(C_R\), for the category of left, respectively, right, \(R\)-modules with the property describing the modules in \(C\).

Now we consider the functor \(\otimes_R\) : \(\mathcal{M}_R \times R\mathcal{M} \to A\). For fixed \(M \in \mathcal{M}_R\) and \(N \in R\mathcal{M}\) we define, in the sense of section 2.4:

\[
\text{Tor}_{n}^{GP}(\,\cdot\,; N) := L_{n}^{GP}(\,\cdot\, \otimes_R N) \quad \text{and} \quad \text{Tor}_{n}^{GF}(M, \,\cdot\,) := L_{n}^{GF}(M \otimes_R \,\cdot\, ),
\]

for all \(N \in R\mathcal{M}\). Together with

\[
\text{Tor}_{n}^{\mathcal{GP}}(\,\cdot\,; N) := L_{n}^{\mathcal{GP}}(\,\cdot\, \otimes_R N) \quad \text{and} \quad \text{Tor}_{n}^{\mathcal{GF}}(M, \,\cdot\,) := L_{n}^{\mathcal{GF}}(M \otimes_R \,\cdot\, ),
\]

the first two \(\text{Tors}\) use proper left Gorenstein projective resolutions, and the last two \(\text{Tors}\) use proper left Gorenstein flat resolutions. In order to compare these different \(\text{Tors}\), we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of

\((X, \,\cdot\,) = (\mathcal{GP}_R, \mathcal{GP}_R) \quad \text{or} \quad (\mathcal{GP}_R, \mathcal{GF}_R),\)

and

\((Y, \,\cdot\,) = (\mathcal{GP}_R, \mathcal{GF}_R) \quad \text{or} \quad (\mathcal{GF}_R, \mathcal{GF}_R),\)

namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[\text{LeftFPD}(R) = \sup\{\text{pd}_RM \mid M \text{ is a left } R\text{-module with } \text{pd}_RM < \infty\}.\]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 4.2.** When \(R\) is commutative and Noetherian, the dimensions \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) coincide and are equal to the Krull dimension of \(R\), by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If \(R\) is right coherent with finite \(\text{LeftFPD}(R)\), then every Gorenstein projective left \(R\)-module is also Gorenstein flat. That is, there is an inclusion \(\mathcal{GP}_R \subseteq \mathcal{GF}_R\).

**Theorem 4.4.** For any left \(R\)-module \(M\), we consider the following three conditions:

(i) The left \(R\)-module \(M\) is \(G\)-flat.

(ii) The Pontryagin dual \(\hat{\text{Hom}}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})\) (which is a right \(R\)-module) is \(G\)-injective.

(iii) \(M\) has an augmented proper right resolution \(0 \to M \to F^0 \to F^1 \to \cdots\) consisting of flat left \(R\)-modules, and \(\text{Tor}_i^R(I, M) = 0\) for all injective right \(R\)-modules \(I\), and all \(i > 0\).

The implication (i) \(\Rightarrow\) (ii) always holds. If \(R\) is right coherent, then also (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i), and hence all three conditions are equivalent.
Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\LeftFPD(R)$ and $\RightFPD(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\Gpd_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R_GP$-resolution of $M$ (which exists by Proposition 4.1). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\RightFPD(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \Hom_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\Hom_R(G^+, H) \cong \Hom_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.1. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite $\LeftFPD(R)$. Let $M$ be a left $R$-module with $\Gfd_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R_GF$-resolution of $M$ (which exists by Proposition 4.2 since $R$ is right coherent). Then the following conclusions hold:

(i) $\Hom_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\LeftFPD(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\Gfd_R M < \infty$ and $R$ is right coherent, Proposition 4.2 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' = \cdots \to G_1 \to G_0 \to M \to 0$ is an $R_GF$-precover of $M$, and $\Gfd_R K' < \infty$. Since $R$ has $\LeftFPD(R) < \infty$, Proposition 6] implies that also $\pd_R K' < \infty$. Now the proof of Lemma 3.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \Hom_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\Hom_R(G^+, H) \cong \Hom_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is right coherent, and that both $\LeftFPD(R)$ and $\RightFPD(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[ \text{Tor}_n^{\mathcal{G}_R}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N). \]

(ii) If $\text{Gpd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[ \text{Tor}_n^{\mathcal{G}_P}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N) \cong \text{Tor}_n^{\mathcal{G}_P}(M, N). \]

(iii) If $\text{Gfd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[ \text{Tor}_n^{\mathcal{G}_F}(M, N) \cong \text{Tor}_n^{\mathcal{G}_P}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N). \]

(iv) If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[ \text{Tor}_n^{\mathcal{G}_P}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N) \cong \text{Tor}_n^{\mathcal{G}_P}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N). \]

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \qed

4.9 (Definition of $g\text{Tor}$ and $\text{GTor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then we write
\[ g\text{Tor}_n^R(M, N) := \text{Tor}_n^{\mathcal{G}_F}(M, N) \cong \text{Tor}_n^{\mathcal{G}_P}(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then we write
\[ \text{GTor}_n^R(M, N) := \text{Tor}_n^{\mathcal{G}_P}(M, N) \cong \text{Tor}_n^{\mathcal{G}_F}(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_R M$, and for every left $R$-module $N$ with $\text{Gpd}_R N < \infty$, we have isomorphisms:
\[ g\text{Tor}_n^R(M, N) \cong \text{GTor}_n^R(M, N) \]
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $\text{GTor}$) with the usual $\text{Tor}$.

**Theorem 4.11.** Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_R M < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_R N < \infty$. If either $\text{fd}_R M < \infty$ or $\text{fd}_R N < \infty$, then there are isomorphisms
\[ g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N) \]
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_R M < \infty$, then we also have $\text{pd}_R M < \infty$ by [13 Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 3.3, $P$ is also a proper left $\mathcal{G}_P R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
\[ g\text{Tor}_n^R(M, N) = \text{Tor}_n^{\mathcal{G}_P}(M, N) = H_n(P \otimes_R N) = \text{Tor}_n^R(M, N), \]
as desired. \qed
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