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GORENSTEIN DERIVED FUNCTORS

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ABSTRACT. Over any associative ring \( R \) it is standard to derive \( \text{Hom}_R(-,-) \) using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains \( \text{Ext}^n_R(-,-) \) in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product \(- \otimes_R -\) using Gorenstein flat modules.

1. INTRODUCTION

When \( R \) is a two-sided Noetherian ring, Auslander and Bridger \cite{AuslanderBridger} introduced in 1969 the G-dimension, \( \text{G-dim}_R M \), for every finite (that is, finitely generated) \( R \)-module \( M \). They proved the inequality \( \text{G-dim}_R M \leq \text{pd}_R M \), with equality \( \text{G-dim}_R M = \text{pd}_R M \) when \( \text{pd}_R M < \infty \), along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring \( R \), Enochs and Jenda in \cite{EnochsJenda} defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if \( R \) is two-sided Noetherian, and \( G \) is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following \cite[Theorem (4.2.6)]{AvramovBuchweitzMartsinkovskyReiten}. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary \( R \)-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- \( R \) is an associative ring. All modules are—if not specified otherwise—left \( R \)-modules, and the category of all \( R \)-modules is denoted \( \mathcal{M} \). We use \( \mathcal{A} \) for the category of abelian groups (that is, \( \mathbb{Z} \)-modules).
- We use \( \mathcal{GP}, \mathcal{GI} \) and \( \mathcal{GF} \) for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat \( R \)-modules; please see \cite{EnochsJenda} and \cite{GoreNSTEIN}, or Definition 2.7 below.
- Furthermore, for each \( R \)-module \( M \) we write \( \text{Gpd}_R M, \text{Gid}_R M \) and \( \text{Gfd}_R M \) for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of \( M \), respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}_R^n(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}_G^n(\cdot, \cdot)$ and $\text{Ext}_G^n(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $GP$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}_G^n(M, N) := \text{H}^n(\text{Hom}_R(G, N)).$$

From [2.1] it will follow that $\text{Ext}_G^n(\cdot, \cdot)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(GP)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $GP$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}_G^n(M', \cdot)$, which is defined on the full subcategory, $\text{RightRes}_M(GI)$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $GI$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}_G^n(M, N) \cong \text{Ext}_G^n(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(GP)$ and $N \in \text{RightRes}_M(GI)$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}_G^n(M, N) \cong \text{Ext}_G^n(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9] Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12] Proposition 2.18] (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(GP)$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $GP$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $GI$-resolution. So $\text{RightRes}_M(GI)$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\text{Ext}_G^n(M, N) \cong \text{Ext}_G^n(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\cdot \otimes_R \cdot$, using proper left $GP$-resolutions and proper left $GF$-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A **proper left \( \mathcal{X} \)-resolution** of \( M \in \mathcal{C} \) is a complex \( X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \rightarrow M \), such that \( X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) is also a complex, and such that the sequence

\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \rightarrow \text{Hom}_\mathcal{C}(X, X_1) \rightarrow \text{Hom}_\mathcal{C}(X, X_0) \rightarrow \text{Hom}_\mathcal{C}(X, M) \rightarrow 0
\]

is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \) as an **augmented** proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \).

**Proper right \( \mathcal{X} \)-resolutions** are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_\mathcal{C}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let \( f: M \rightarrow M' \) be a morphism in \( \mathcal{C} \), and consider the diagram

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\cdots & \rightarrow & X'_2 & \rightarrow & X'_1 & \rightarrow & X'_0 & \rightarrow & M' & \rightarrow & 0
\end{array}
\]

where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n: X_n \rightarrow X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).

(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2]. \( \square \)

**Remark 2.3.** A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi: X \rightarrow M \), where \( X \in \mathcal{X} \), such that the sequence

\[
\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', M) \rightarrow 0
\]

is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1}) \), \( i > 0 \), and \( X_0 \rightarrow M \) are \( \mathcal{X} \)-precovers.

- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor \( T: \mathcal{C} \rightarrow \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n \)th left derived functor

\[
L_n^\mathcal{X}T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \rightarrow \mathcal{E}
\]
applied to 

If furthermore any proper left \( X \)-resolution of \( M \in \text{LeftRes}_C(\mathcal{X}) \). Similarly, the \( n^{th} \) right derived functor

\[
R^n_X T: \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}
\]

of \( T \) with respect to \( \mathcal{X} \) is defined by \( R^n_X T(N) = H_n(T(Y)) \), where \( Y \) is any proper right \( \mathcal{X} \)-resolution of \( N \in \text{RightRes}_C(\mathcal{X}) \). These constructions are well-defined and functorial in the arguments \( M \) and \( N \) by Proposition 2.2.

The situation where \( T \) is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category \( D \), together with a full subcategory \( \mathcal{Y} \subseteq D \) and an additive functor \( F: C \times D \to \mathcal{E} \) in two variables. We will assume that \( F \) is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of \( F \) is not important, and the definitions and results below can easily be modified to fit the situation where \( F \) is covariant in both variables, say.

For fixed \( M \in C \) and \( N \in D \) we can then consider the two right derived functors as in [2.4]

\[
R^n_X F(-, N): \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^n_Y F(M, -): \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.
\]

If furthermore \( M \in \text{LeftRes}_C(\mathcal{X}) \) and \( N \in \text{RightRes}_D(\mathcal{Y}) \), we can ask for a sufficient condition to ensure that

\[
R^n_X F(M, N) \cong R^n_Y F(M, N),
\]

functorial in \( M \) and \( N \). Here we wrote \( R^n_X F(M, N) \) for the functor \( R^n_X F(-, N) \) applied to \( M \). Another, and perhaps better, notation could be

\[
R^n_X F(-, N)[M].
\]

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor \( F: C \times D \to \mathcal{E} \) which is contravariant in the first variable and covariant in the second variable, together with the full subcategories \( \mathcal{X} \subseteq C \) and \( \mathcal{Y} \subseteq D \). Assume that we have full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \text{LeftRes}_C(\mathcal{X}) \) and \( \text{RightRes}_D(\mathcal{Y}) \), respectively, satisfying:

(i) \( \mathcal{X} \subseteq \mathcal{X} \) and \( \mathcal{Y} \subseteq \mathcal{Y} \).

(ii) Every \( M \in \mathcal{X} \) has an augmented proper left \( \mathcal{X} \)-resolution \( \cdots \to X_1 \to X_0 \to M \to 0 \), such that \( 0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots \) is exact for all \( Y \in \mathcal{Y} \).

(iii) Every \( N \in \mathcal{Y} \) has an augmented proper right \( \mathcal{Y} \)-resolution \( 0 \to N \to Y^0 \to Y^1 \to \cdots \), such that \( 0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots \) is exact for all \( X \in \mathcal{X} \).

Then we have functorial isomorphisms

\[
R^n_X F(M, N) \cong R^n_Y F(M, N),
\]

for all \( M \in \mathcal{X} \) and \( N \in \mathcal{Y} \).
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [5, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories \( \mathcal{X}, \mathcal{X}, \mathcal{C} \) and \( \mathcal{Y}, \mathcal{Y}, \mathcal{D} \), including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

\[
P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,
\]

such that \( \text{Hom}_R(P, Q) \) is exact for every projective \( R \)-module \( Q \). An \( R \)-module \( M \) is called Gorenstein projective (\( G \)-projective for short), if there exists a complete projective resolution \( P \) with \( M \cong \text{Im}(P_0 \to P_{-1}) \). Gorenstein injective (\( G \)-injective for short) modules are defined dually.

3. Gorenstein derivings \( \text{Hom}_R(-,-) \)

We now return to categories of modules. We use \( \widetilde{\mathcal{P}}, \widetilde{\mathcal{I}} \) and \( \widetilde{\mathcal{F}} \) to denote the class of \( R \)-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, \( \mathcal{P} \)-precovers are always surjective, and \( \widetilde{\mathcal{P}} \) contains all modules with finite projective dimension.

We now consider the functor \( \text{Hom}_R(-,-): \mathcal{M} \times \mathcal{M} \to \mathcal{A} \), together with the categories

\[
\mathcal{X} = \mathcal{G\mathcal{P}}, \quad \mathcal{X} = \widetilde{\mathcal{G\mathcal{P}}} \quad \text{and} \quad \mathcal{Y} = \mathcal{G\mathcal{I}}, \quad \mathcal{Y} = \widetilde{\mathcal{G\mathcal{I}}}.
\]

In this case we define, in the sense of section 2.1

\[
\text{Ext}^n_{\mathcal{G\mathcal{P}}}(\cdot, \cdot) = R^n_{\mathcal{G\mathcal{P}}} \text{Hom}_R(\cdot, \cdot) \quad \text{and} \quad \text{Ext}^n_{\mathcal{G\mathcal{I}}}(M, -) = R^n_{\mathcal{G\mathcal{I}}} \text{Hom}_R(M, -),
\]

for fixed \( R \)-modules \( M \) and \( N \). We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If \( M \) is an \( R \)-module with \( \text{Gpd}_R M < \infty \), then there exists a short exact sequence \( 0 \to K \to G \to M \to 0 \), where \( G \to M \) is a \( \mathcal{G\mathcal{P}} \)-precover of \( M \) (please see Remark 2.23), and \( \text{pd}_R K = \text{Gpd}_R M - 1 \) (in the case where \( M \) is Gorenstein projective, this should be interpreted as \( K = 0 \)).

Consequently, every \( R \)-module with finite Gorenstein projective dimension has a proper left \( \mathcal{G\mathcal{P}} \)-resolution (that is, there is an inclusion \( \widetilde{\mathcal{G\mathcal{P}}} \subseteq \text{LeftRes}_M(\mathcal{G\mathcal{P}}) \)).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let \( M \) be any \( R \)-module with \( \text{Gpd}_R M < \infty \). Then

\[
\text{Gpd}_R M = \text{sup}\{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R \text{-module } L \text{ with } \text{pd}_R L < \infty\}.
\]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an \( R \)-module \( M \) is given by

\[
\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \}.
\]

It also follows that if \( \text{pd}_R M < \infty \), then every projective resolution of \( M \) is actually a proper left \( GP \)-resolution of \( M \).

Lemma 3.4. Assume that \( M \) is an \( R \)-module with finite Gorenstein projective dimension, and let \( G^+ = \cdots \to G_1 \to G_0 \to M \to 0 \) be an augmented proper left \( GP \)-resolution of \( M \) (which exists by Proposition 3.1). Then \( \text{Hom}_R(G^+, H) \) is exact for all Gorenstein injective modules \( H \).

Proof. We split the proper resolution \( G^+ \) into short exact sequences. Hence it suffices to show exactness of \( \text{Hom}_R(S, H) \) for all Gorenstein injective modules \( H \) and all short exact sequences

\[
S = 0 \to K \to G \to M \to 0,
\]

where \( G \to M \) is a \( GP \)-precover of some module \( M \) with \( \text{Gpd}_R M < \infty \) (recall that \( GP \)-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,

\[
S' = 0 \to K' \xrightarrow{i} G' \xrightarrow{\pi} M \xrightarrow{\epsilon} 0,
\]

where \( \pi: G' \to M \) is a \( GP \)-precover and \( \text{pd}_R K' < \infty \).

It is easy to see (as in Proposition 3.1) that the complexes \( S \) and \( S' \) are homotopy equivalent, and thus so are the complexes \( \text{Hom}_R(S, H) \) and \( \text{Hom}_R(S', H) \) for every (Gorenstein injective) module \( H \). Hence it suffices to show the exactness of \( \text{Hom}_R(S', H) \) whenever \( H \) is Gorenstein injective.

Now let \( H \) be any Gorenstein injective module. We need to prove the exactness of

\[
\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \xrightarrow{\epsilon} 0.
\]

To see this, let \( \alpha: K' \to H \) be any homomorphism. We wish to find \( \varphi: G' \to H \) such that \( \varphi \iota = \alpha \). Now pick an exact sequence

\[
0 \to \tilde{H} \to E \xrightarrow{g} H \to 0,
\]

where \( E \) is injective, and \( \tilde{H} \) is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines \( H \)). Since \( \tilde{H} \) is Gorenstein injective and \( \text{pd}_R K' < \infty \), we get \( \text{Ext}_R^1(K', \tilde{H}) = 0 \) by [1, Lemma 1.3], and thus a lifting \( \varepsilon: K' \to E \) with \( g \varepsilon = \alpha \):

Next, injectivity of \( E \) gives \( \tilde{\varepsilon}: G' \to E \) with \( \tilde{\varepsilon} \iota = \varepsilon \). Now \( \varphi = g \tilde{\varepsilon} : G' \to H \) is the desired map. \( \square \)

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $GI$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$.

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N),$$

which are functorial in $M$ and $N$. \hfill \Box

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G^2}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(1) $\text{pd}_R M < \infty$ or (2) $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N < \infty$.

(ii) There are natural isomorphisms $\text{Ext}^n_{G^2}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(3) $\text{id}_R N < \infty$ or (4) $N \in \text{RightRes}_M(GT)$ and $\text{pd}_R M < \infty$.

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then

$$\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$$

is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $GP$-resolution of $M$, and thus

$$\text{Ext}^n_{GP}(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{m+i}_R(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A] implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt$^n_R(-, -)$.

\hfill \Box
4. Gorenstein deriving $- \otimes_R -$ 

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}$, $\mathcal{GP}$, etc.), we write $\mathcal{R} \mathcal{C}$, respectively, $\mathcal{C} \mathcal{R}$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $\mathcal{R}^G$: $\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}$ and $N \in \mathcal{R}M$ we define, in the sense of section 2.4:

$\text{Tor}^n_{\mathcal{GP}}(\mathcal{-} \otimes_R N) := L^n_{\mathcal{GP}}(\mathcal{-} \otimes_R N)$ and $\text{Tor}^n_{\mathcal{GP}}(M, \mathcal{-}) := L^n_{\mathcal{GP}}(M \otimes_R -)$,

together with

$\text{Tor}^n_{\mathcal{GF}}(\mathcal{-} \otimes_R N) := L^n_{\mathcal{GF}}(\mathcal{-} \otimes_R N)$ and $\text{Tor}^n_{\mathcal{GF}}(M, \mathcal{-}) := L^n_{\mathcal{GF}}(M \otimes_R -)$.

The first two $\text{Tor}$s use proper left Gorenstein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of

$(X, \tilde{X}) = (\mathcal{GP}_R, \mathcal{GP}_R)$ or $(\mathcal{GF}_R, \mathcal{GF}_R)$,

and

$(Y, \tilde{Y}) = (\mathcal{GP}_R, \mathcal{GP}_R)$ or $(\mathcal{GF}_R, \mathcal{GF}_R)$,

namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$\text{LeftFPD}(R) = \text{sup}(\text{pd}_R M | M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty)$.

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathcal{GP}_R \subseteq \mathcal{GF}_R$. $\square$

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent. $\square$
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}$F-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}$F-resolution (that is, there is an inclusion $R\mathcal{G}$F $\subseteq \text{LeftRes}_{R,M}(R\mathcal{G}$F$)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}P$-resolution of $M$ (which exists by Proposition [4.4]). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem [4.4] above, the Pontryagin dual $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition [3.1]. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition [4.3] implies that every Gorenstein projective right $R$-module also is Gorenstein flat. □

Lemma 4.7. Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition [4.4] since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition [4.5] gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}$F-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma [3.1] applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem [4.4] above. By the result (ii), just proved, we have exactness of

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition [4.3] implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii). □

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N).
\]

(ii) If \( \text{Gpd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N).
\]

(iii) If \( \text{Gfd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N).
\]

(iv) If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N).
\]

All the isomorphisms are functorial in \( M \) and \( N \).

**Proof.** Use Lemmas [4.6] and [4.7] as input in the covariant-covariant version of Theorem 2.6. \( \square \)

### 4.9 (Definition of \( g\text{Tor} \) and \( G\text{Tor} \))

Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module, and let \( N \) be a left \( R \)-module. If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then we write
\[
g\text{Tor}_n^R(M, N) := \Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(i). If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then we write
\[
G\text{Tor}_n^R(M, N) := \Tor_n^{G\mathcal{F}}(M, N) \cong \Tor_n^{G\mathcal{F}}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. For every right \( R \)-module \( M \) with finite \( \text{Gpd}_R M \), and for every left \( R \)-module \( N \) with \( \text{Gpd}_R N < \infty \), we have isomorphisms:
\[
g\text{Tor}_n^R(M, N) \cong G\text{Tor}_n^R(M, N)
\]
that are functorial in \( M \) and \( N \).

Finally we compare \( g\text{Tor} \) (and hence \( G\text{Tor} \)) with the usual \( \text{Tor} \).

**Theorem 4.11.** Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module with \( \text{Gfd}_R M < \infty \), and let \( N \) be a left \( R \)-module with \( \text{Gfd}_R N < \infty \). If either \( \text{fd}_R M < \infty \) or \( \text{pd}_R N < \infty \), then there are isomorphisms
\[
g\text{Tor}_n^R(M, N) \cong \Tor_n^R(M, N)
\]
that are functorial in \( M \) and \( N \).

**Proof.** If \( \text{fd}_R M < \infty \), then we also have \( \text{pd}_R M < \infty \) by [13] Proposition 6] (since \( \text{RightFPD}(R) < \infty \)). Let \( \mathcal{P} \) be any projective resolution of \( M \). As noted in Remark 3.3, \( \mathcal{P} \) is also a proper left \( \mathcal{GP} \) resolution of \( M \). Hence, Theorem 4.8 ii) and the definitions give
\[
g\text{Tor}_n^R(M, N) = \Tor_n^{G\mathcal{P}}(M, N) = H_n(\mathcal{P} \otimes_R N) = \Tor_n^R(M, N),
\]
as desired. \( \square \)
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References


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