On the classification of nonsimple graph C*-algebras

Eilers, Søren; Tomforde, Mark

Published in:
Mathematische Annalen

Publication date:
2010

Document Version
Early version, also known as pre-print

Citation for published version (APA):
ON THE CLASSIFICATION OF NONSIMPLE GRAPH
$C^*$-ALGEBRAS

SØREN EILERS AND MARK TOMFORDE

Abstract. We prove that a graph $C^*$-algebra with exactly one proper nontrivial ideal is classified up to stable isomorphism by its associated six-term exact sequence in $K$-theory. We prove that a similar classification also holds for a graph $C^*$-algebra with a largest proper ideal that is an AF-algebra. Our results are based on a general method developed by the first named author with Restorff and Ruiz. As a key step in the argument, we show how to produce stability for certain full hereditary subalgebras associated to such graph $C^*$-algebras. We further prove that, except under trivial circumstances, a unique proper nontrivial ideal in a graph $C^*$-algebra is stable.

1. Introduction

The classification program for $C^*$-algebras has for the most part progressed independently for the classes of infinite and finite $C^*$-algebras. Great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered $K_0$-group. In the infinite case, there are a number of results for purely infinite $C^*$-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite $C^*$-algebras up to stable isomorphism by the $K_0$-group together with the $K_1$-group. For nonsimple purely infinite $C^*$-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite $C^*$-algebras containing exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of $K$-groups [Rør97]. Restorff has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtrated $K$-theory [Res06, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite $C^*$-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated $K$-theory [MN, Theorem 4.14]. However, in all of these situations the nonsimple $C^*$-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.
Recently, the first named author with Restorff and Ruiz have provided a framework for classifying nonsimple $C^*$-algebras that are not necessarily AF-algebras or purely infinite $C^*$-algebras. In particular, these authors have shown in [ERR] that certain extensions of classifiable $C^*$-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in $K$-theory. This has allowed for the classification of certain nonsimple $C^*$-algebras in which there are ideals and quotients of mixed type (some finite and some infinite).

In this paper we consider the classification of nonsimple graph $C^*$-algebras. Simple graph $C^*$-algebras are known to be either AF-algebras or purely infinite algebras, and thus are classified by their $K$-groups according to either Elliott’s Theorem or the Kirchberg-Phillips Theorem. Therefore, we begin by considering nonsimple graph $C^*$-algebras with exactly one proper nontrivial ideal. These $C^*$-algebras will be extensions of simple $C^*$-algebras that are AF or purely infinite by other simple $C^*$-algebras that are AF or purely infinite — with mixing of the types allowed. These nonsimple graph $C^*$-algebras are similar to the extensions considered in [ERR], however, the results of [ERR] do not apply directly. Instead, we must do a fair bit of work, using the techniques from the theory of graph $C^*$-algebras, to show that the machinery of [ERR] can be used to classify these extensions; it is verifying the requirement of fullness that is most difficult in this context. Ultimately, however, we are able to show that a graph $C^*$-algebra with exactly one proper nontrivial ideal is classified up to stable isomorphism by the six-term exact sequence in $K$-theory of the corresponding extension. Additionally, we are able to show that a graph $C^*$-algebra with a largest proper ideal that is an AF-algebra is also classified up to stable isomorphism by the six-term exact sequence in $K$-theory of the corresponding extension.

It is also worthwhile to note that the extensions of graph $C^*$-algebras classified in this paper constitute a very large class. Every AF-algebra is stably isomorphic to a graph $C^*$-algebra, and every Kirchberg algebra with free $K_1$-group is stably isomorphic to a a graph $C^*$-algebra. Thus the extensions we consider comprise a wide variety of extensions of AF-algebras (respectively, purely infinite algebras) by purely infinite algebras (respectively, AF-algebras).

While there is little hope to generalize the methods in [ERR] to general (even finite) ideal lattices in a context covering all graph $C^*$-algebras, the classifications we obtain in this paper suggest that a complete classification of graph $C^*$-algebras generalizing the Cuntz-Krieger case solved in [Res06] may be possible by other methods. Such a result may involve generalizing the work of Boyle and Huang to graph $C^*$-algebras and mimicking the approach used by Restorff; or perhaps it may be accomplished by generalizing Kirchberg’s isomorphism theorem to allow for subquotients which are AF-algebras, and then in this special case (probably using the global vanishing of one connecting map of $K$-theory) overcoming the difficulties of projective dimension of the invariants exposed by Meyer and Nest. Neither of these
approaches seem within immediate reach, but both appear plausible to be successful at some future stage. It is also an open problem, and possibly much less difficult, to establish isomorphism directly between unital graph $C^*$-algebras by keeping track of the class of the unit in the $K_0$-groups. We mention that the methods of [RR07] do not seem to generalize to this setting.

This paper is organized as follows. In §2 we establish notation and conventions for graph $C^*$-algebras and extensions. In §3 we derive a number of preliminary results for graph $C^*$-algebras with the goal of applying the methods of [ERR]. In §4 we use our results from §3 and the results of [ERR] to prove our two main theorems: In Theorem 4.5 we show that if $A$ is a graph $C^*$-algebra with exactly one proper nontrivial ideal $I$, then $A$ is classified up to stable isomorphism by the six-term exact sequence in $K$-theory coming from the extension $0 \to I \to A \to A/I \to 0$. In Theorem 4.7 we show that if $A$ is a graph $C^*$-algebra with a largest proper ideal $I$ that is an AF-algebra, then $A$ is classified up to stable isomorphism by the six-term exact sequence in $K$-theory coming from the extension $0 \to I \to A \to A/I \to 0$. In §5 we consider a variety of examples, and also use our results to classify the stable isomorphism classes of the $C^*$-algebras of all graphs having exactly two vertices and satisfying Condition (K). (Be aware that although these graphs have only two vertices, the graphs are allowed to contain a finite or countably infinite number of edges.) We find that even for this small collection of graphs, the associated $C^*$-algebras fall into a variety of stable isomorphism classes, and there are quite a few cases to consider. We conclude in §6 by proving that if $A$ is a graph $C^*$-algebra that is not a nonunital AF-algebra, and if $A$ contains a unique proper nontrivial ideal $I$, then $I$ is stable.

2. Notation and conventions

We establish some basic facts and notation for graph $C^*$-algebras and extensions.

2.1. Notation and conventions for graph $C^*$-algebra. A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a sink if $|s^{-1}(v)| = 0$, and $v$ is called an infinite emitter if $|s^{-1}(v)| = \infty$. A graph $E$ is said to be row-finite if it has no infinite emitters. If $v$ is either a sink or an infinite emitter, then we call $v$ a singular vertex. We write $E^0_{\text{sing}}$ for the set of singular vertices. Vertices that are not singular vertices are called regular vertices and we write $E^0_{\text{reg}}$ for the set of regular vertices. For any graph $E$, the vertex matrix is the $E^0 \times E^0$ matrix $A_E$ with $A_E(v, w) := |\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}|$. Note that the entries of $A_E$ are elements of $\{0, 1, 2, \ldots\} \cup \{\infty\}$.

If $E$ is a graph, a Cuntz-Krieger $E$-family is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the Cuntz-Krieger relations:

1. $s_e^*s_e = p_{r(e)}$ for every $e \in E^1$;
(2) \( s_es^*_e \leq p_{s(e)} \) for every \( e \in E^1 \);
(3) \( p_v = \sum_{s(e) = v} s_es^*_e \) for every \( v \in E^0 \) that is not a singular vertex.

The graph algebra \( C^*(E) \) is defined to be the \( C^* \)-algebra generated by a universal Cuntz-Krieger \( E \)-family.

A path in \( E \) is a sequence of edges \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_n \) with \( r(\alpha_i) = s(\alpha_{i+1}) \) for \( 1 \leq i < n \), and we say that \( \alpha \) has length \( |\alpha| = n \). We let \( E^n \) denote the set of all paths of length \( n \), and we let \( E^* := \bigcup_{n=0}^{\infty} E^n \) denote the set of finite paths in \( G \). Note that vertices are considered paths of length zero. The maps \( r, s \) extend to \( E^* \), and for \( v, w \in E^0 \) we write \( v \geq w \) if there exists a path \( \alpha \in E^* \) with \( s(\alpha) = v \) and \( r(\alpha) = w \). Also for a path \( \alpha := \alpha_1 \ldots \alpha_n \) we define \( s_\alpha := s_{\alpha_1} \ldots s_{\alpha_n} \), and for a vertex \( v \in E^0 \) we let \( s_v := p_v \). It is a consequence of the Cuntz-Krieger relations that \( C^*(E) = \text{span}\{s_\alpha s^*_\beta : \alpha, \beta \in E^* \} \) and \( r(\alpha) = r(\beta) \).

We say that a path \( \alpha := \alpha_1 \ldots \alpha_n \) of length 1 or greater is a cycle if \( r(\alpha) = s(\alpha) \), and we call the vertex \( s(\alpha) = r(\alpha) \) the base point of the cycle. A cycle is said to be simple if \( s(\alpha_i) \neq s(\alpha_1) \) for all \( 1 < i \leq n \). The following is an important condition in the theory of graph \( C^* \)-algebras.

**Condition (K):** No vertex in \( E \) is the base point of exactly one simple cycle; that is, every vertex is either the base point of no cycles or at least two simple cycles.

For any graph \( E \) a subset \( H \subseteq E^0 \) is hereditary if whenever \( v, w \in E^0 \) with \( v \in H \) and \( v \geq w \), then \( w \in H \). A hereditary subset \( H \) is saturated if whenever \( v \in E^0_{\text{reg}} \) with \( r(s^{-1}(v)) \subseteq H \), then \( v \in H \). For any saturated hereditary subset \( H \), the breaking vertices corresponding to \( H \) are the elements of the set

\[ B_H := \{ v \in E^0 : \|s^{-1}(v)\| = \infty \text{ and } 0 < \|s^{-1}(v) \cap r^{-1}(E^0 \setminus H)\| < \infty \}. \]

An admissible pair \((H, S)\) consists of a saturated hereditary subset \( H \) and a subset \( S \subseteq B_H \). For a fixed graph \( E \) we order the collection of admissible pairs for \( E \) by defining \((H, S) \leq (H', S')\) if and only if \( H \subseteq H' \) and \( S \subseteq H' \cup S' \). For any admissible pair \((H, S)\) we define

\[ I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p^H_{v_0} : v_0 \in S\}, \]

where \( p^H_{v_0} \) is the gap projection defined by

\[ p^H_{v_0} := p_{v_0} - \sum_{s(e) = v_0, r(e) \notin H} s_es^*_e. \]

Note that the definition of \( B_H \) ensures that the sum on the right is finite.

For any graph \( E \) there is a canonical gauge action \( \gamma : T \to \text{Aut } C^*(E) \) with the property that for any \( z \in T \) we have \( \gamma_z(p_v) = p_v \) for all \( v \in E^0 \) and \( \gamma_z(s_e) = zs_e \) for all \( e \in E^1 \). We say that an ideal \( I \triangleleft C^*(E) \) is gauge invariant if \( \gamma_z(I) \subseteq I \) for all \( z \in T \).
There is a bijective correspondence between the lattice of admissible pairs of $E$ and the lattice of gauge-invariant ideals of $C^*(E)$ given by $(H, S) \mapsto I_{(H,S)}$ [BHRS, Theorem 3.6]. When $E$ satisfies Condition (K), all ideals of $C^*(E)$ are gauge invariant [Tom06, Theorem 2.24] and the map $(H, S) \mapsto I_{(H,S)}$ is onto the lattice of ideals of $C^*(E)$. When $BH = \emptyset$, we write $I_H$ in place of $I_{(H,\emptyset)}$ and observe that $I_H$ equals the ideal generated by $\{pv : v \in H\}$. Note that if $E$ is row-finite, then $BH$ is empty for every saturated hereditary subset $H$.

2.2. Notation and conventions for extensions. All ideals in $C^*$-algebras will be considered to be closed two-sided ideals. An element $a$ of a $C^*$-algebra $A$ (respectively, a subset $S \subseteq A$) is said to be full if $a$ (respectively, $S$) is not contained in any proper ideal of $A$. A map $\phi : A \to B$ is full if $\text{im } \phi$ is full in $B$.

If $A$ and $B$ are $C^*$-algebras, an extension of $A$ by $B$ consists of a $C^*$-algebra $E$ and a short exact sequence

$$0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0.$$ 

We say that the extension $\epsilon$ is essential if $\alpha(B)$ is an essential ideal of $E$, and we say that the extension $\epsilon$ is unital if $E$ is a unital $C^*$-algebra. For any extension there exist unique $*$-homomorphisms $\eta_\epsilon : E \to \mathcal{M}(B)$ and $\tau_\epsilon : A \to \mathcal{Q}(B) := \mathcal{M}(B)/B$ which make the diagram

$$
\begin{array}{c}
0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to B \xrightarrow{i} \mathcal{M}(B) \xrightarrow{\pi} \mathcal{Q}(B) \to 0
\end{array}
$$

commute. The $*$-homomorphism $\tau_\epsilon$ is called the Busby invariant of the extension, and the extension is essential if and only if $\tau_\epsilon$ is injective. An extension $\epsilon$ is full if the associated Busby invariant $\tau_\epsilon$ has the property that $\tau_\epsilon(a)$ is full in $\mathcal{Q}(A)$ for every $a \in A \setminus \{0\}$.

For an extension $\epsilon$, we let $K_{\text{six}}(\epsilon)$ denote the cyclic six-term exact sequence of $K$-groups

$$
\begin{array}{c}
K_0(B) \xrightarrow{\alpha} K_0(E) \xrightarrow{\beta} K_0(A) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_1(A) \xleftarrow{\alpha} K_1(E) \xleftarrow{\beta} K_1(B)
\end{array}
$$

where $K_0(B)$, $K_0(E)$, and $K_0(A)$ are viewed as (pre-)ordered groups. Given two extensions

$$
\begin{array}{c}
\epsilon_1 : \\
0 \to B_1 \xrightarrow{\alpha_1} E_1 \xrightarrow{\beta_1} A_1 \to 0 \\
\epsilon_2 : \\
0 \to B_2 \xrightarrow{\alpha_2} E_2 \xrightarrow{\beta_2} A_2 \to 0
\end{array}
$$
we say \( K_{s\xi}(e_1) \) is isomorphic to \( K_{s\xi}(e_2) \), written \( K_{s\xi}(e_1) \cong K_{s\xi}(e_2) \), if there exist isomorphisms \( \alpha, \beta, \gamma, \delta, \epsilon, \) and \( \zeta \) making the following diagram commute:

\[
\begin{array}{ccc}
K_0(B_1) & \rightarrow & K_0(E_1) \\
\downarrow \alpha & & \downarrow \beta \\
K_0(B_2) & \rightarrow & K_0(E_2) & \rightarrow & K_0(A_2) \\
\downarrow \gamma & & \downarrow \delta \\
K_1(A_2) & \leftarrow & K_1(E_2) & \leftarrow & K_1(B_2) \\
\downarrow \epsilon & & \downarrow \zeta \\
K_1(A_1) & \leftarrow & K_1(E_1) & \leftarrow & K_1(B_1)
\end{array}
\]

and where \( \alpha, \beta, \) and \( \gamma \) are isomorphisms of (pre-)ordered groups.

3. Preliminary graph \( C^* \)-algebra results

In this section we develop a few results for graph \( C^* \)-algebras in order to apply the methods of \[ERR\] in \[4\]. However, several of these results are interesting in their own right.

**Lemma 3.1.** If \( E \) is a graph such that \( C^*(E) \) contains a unique proper nontrivial ideal \( I \), then the following six conditions are satisfied:

1. \( E \) satisfies Condition (K),
2. \( E \) contains exactly three saturated hereditary subsets \( \{\emptyset, H, E_0\} \),
3. \( E \) contains no breaking vertices; i.e., \( B_H = \emptyset \),
4. \( I \) is a gauge invariant ideal and \( I_H = I \),
5. If \( X \) is a nonempty hereditary subset of \( E \), then \( X \cap H \neq \emptyset \), and
6. \( E \) has at most one sink, and if \( v \) is a sink of \( E \) then \( v \in H \).

**Proof.** Suppose that \( E \) does not satisfy Condition (K). Then by \[Tom06, Proposition 1.17\] there exists a saturated hereditary subset \( H \subseteq E_0 \) such that \( E \setminus H \) contains a cycle \( \alpha = e_1 \ldots e_n \) with no exits. The set \( X = \{s(e_i)\}_{i=1}^{n} \) is a hereditary subset of \( E \setminus H \), and \( I_X \) is an ideal in \( C^*(E \setminus H) \) Morita equivalent to \( M_n(C(T)) \) (see \[BHRS\] Proposition 3.4) and \[Rae05, Example 2.14\] for details). Thus \( I_X \), and hence \( C^*(E \setminus H) \), contains a countably infinite number of ideals. Since \( C^*(E \setminus H) \cong C^*(E)/I_{(H,B_H)} \) \[BHRS\] Proposition 3.4], this implies that \( C^*(E) \) has a countably infinite number of ideals. Hence if \( C^*(E) \) has a finite number of ideals, \( E \) satisfies Condition (K).

Because \( E \) satisfies Condition (K), it follows from \[DT02, Theorem 3.5\] that the ideals of \( C^*(E) \) are in one-to-one correspondence with the pairs \( (H,S) \) where \( H \) is saturated hereditary, and \( S \subseteq B_H \) is a subset of the breaking vertices of \( H \). Since \( E \) contains a unique proper nontrivial ideal, it follows that \( E \) contains a unique saturated hereditary subset \( H \) not equal to
$E^0$ or $\emptyset$, and that there are no breaking vertices; i.e., $B_H = \emptyset$. It must also be the case that $I = I_H$. Moreover, since $E$ satisfies Condition (K), [BHRS, Corollary 3.8] shows that all ideals of $C^*(E)$ are gauge-invariant.

In addition, suppose $X$ is a hereditary subset with $X \cap H = \emptyset$. Since $H$ is hereditary, none of the vertices in $H$ can reach $X$, and thus the saturation $\overline{X}$ contains no vertices of $H$, and $\overline{X} \cap H = \emptyset$. But then $\overline{X}$ is a saturated hereditary subset that does not contain the vertices of $H$, and hence must be equal to $\emptyset$. Thus if $X$ is a nonempty hereditary subset of $E$, then $X \cap H \neq \emptyset$.

Finally, suppose $v$ is a sink of $E$. Consider the hereditary subset $X := \{v\}$. From the previous paragraph it follows that $X \cap H \neq \emptyset$ and hence $v \in H$. In addition, there cannot be a second sink in $E$, for if $v'$ is a sink, then $X := \{v\}$ and $Y := \{v'\}$ are distinct hereditary sets. Since $v$ cannot reach $v'$, we see that $v$ is not in the saturation $\overline{Y}$. Similarly, since $v'$ cannot reach $v$, we have that $v'$ is not in the saturation $\overline{X}$. Thus $\overline{X}$ and $\overline{Y}$ are distinct saturated hereditary subsets of $E$ that are proper and nontrivial, which is a contradiction. It follows that there is at most one sink in $E$. □

**Remark 3.2.** According to [DHS03, Lemma 1.3] and [BHRS, Corollary 3.5] the $C^*$-algebras $I$ and $A/I$ are in this case also graph algebras. As they are necessarily simple, they must be either Kirchberg algebras or AF-algebras. We will denote the four cases thus occurring as follows

<table>
<thead>
<tr>
<th>Case</th>
<th>$I$</th>
<th>$A/I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[11]</td>
<td>AF</td>
<td>AF</td>
</tr>
<tr>
<td>[1\infty]</td>
<td>AF</td>
<td>Kirchberg</td>
</tr>
<tr>
<td>[\infty1]</td>
<td>Kirchberg</td>
<td>AF</td>
</tr>
<tr>
<td>[\infty\infty]</td>
<td>Kirchberg</td>
<td>Kirchberg</td>
</tr>
</tbody>
</table>

**Definition 3.3.** Let $A$ be a $C^*$-algebra. A proper ideal $I \subset A$ is a largest proper ideal of $A$ if whenever $J \subset A$, then either $J \subseteq I$ or $J = A$.

Observe that a largest proper ideal is always an essential ideal. Also note that if $A$ is a $C^*$-algebra with a unique proper nontrivial ideal $I$, then $I$ is a largest proper ideal; and if $A$ is a simple $C^*$-algebra then $\{0\}$ is a largest proper ideal.

**Lemma 3.4.** Let $E$ be a graph, and suppose that $I$ is a largest proper ideal of $C^*(E)$. Then $I$ is gauge invariant and $I = I_{(H,B_H)}$ for some saturated hereditary subset $H$ of $E^0$. Furthermore, if $K$ is any saturated hereditary subset of $E$, then either $K \subseteq H$ or $K = E^0$.

**Proof.** Let $\gamma$ denote the canonical gauge action of $\mathbb{T}$ on $C^*(E)$. For any $z \in \mathbb{T}$ we have that $\gamma_z(I)$ is a proper ideal of $C^*(E)$. Since $I$ is a largest proper ideal of $C^*(E)$, it follows that $\gamma_z(I) \subset I$. A similar argument shows that $\gamma_{z^{-1}}(I) \subseteq I$. Thus $\gamma_z(I) = I$ and $I$ is gauge invariant. It follows from
[BHRS] Theorem 3.6] that \( I = I_{(H,S)} \) for some saturated hereditary subset \( H \) of \( E^0 \) and some subset \( S \subseteq B_H \). Because \( I \) is a largest proper ideal, it follows that \( S = B_H \), and hence \( I = I_{(H,B_H)} \). Furthermore, if \( K \) is a saturated hereditary subset, then either \( I_{(K,B_K)} \subseteq I_{(H,B_H)} \) or \( I_{(K,B_K)} = C^*(E) \). Hence either \( K \subseteq H \) or \( K = E^0 \).

\[ \square \]

**Lemma 3.5.** Let \( E \) be a graph and suppose that \( I \) is a largest proper ideal of \( C^*(E) \) with the property that \( C^*(E)/I \) is purely infinite. Then \( I = I_{(H,B_H)} \) for some saturated hereditary subset \( H \) of \( E^0 \), and there exists a cycle \( \gamma \) in \( E \setminus H \) and an edge \( f \in E^1 \) with \( s(f) = s(\gamma) \) and \( r(f) \in H \). Furthermore, if \( x \in E^0 \), then \( x \geq s(\gamma) \) if and only if \( x \in E^0 \setminus H \).

**Proof.** Lemma 3.4 shows that \( I = I_{(H,B_H)} \) for some saturated hereditary subset \( H \) of \( E^0 \). It follows from [BHRS] Corollary 3.5] that \( C^*(E)/I_{(H,B_H)} \cong C^*(E \setminus H) \), where \( E \setminus H \) is the subgraph of \( E \) with \( (E \setminus H)^0 := E^0 \setminus H \) and \( (E \setminus H)^1 := E^1 \setminus r^{-1}(H) \). Since \( C^*(E \setminus H) \) is purely infinite, it follows from [DT05] Corollary 2.14] that \( E \setminus H \) contains a cycle \( \alpha \). Define \( K := \{ x \in E^0 : x \not\geq s(\alpha) \} \). Then \( K \) is saturated hereditary, \( H \subseteq K \), and \( K \neq E^0 \). Hence \( I_{(H,B_H)} \subseteq I_{(K,B_K)} \neq C^*(E) \), and the fact that \( I_{(H,B_H)} \) is a largest proper ideal implies that \( I_{(H,B_H)} = I_{(K,B_K)} \) so that \( H = K \). Hence for \( x \in E^0 \) we have \( x \geq s(\alpha) \) if and only if \( x \in E^0 \setminus H \).

Consider the set \( J := \{ x \in E^0 : s(\alpha) \geq x \} \). Then \( J \) is a hereditary subset and we let \( J \) denote its saturation. Since \( I_{(H,B_H)} \) is a largest proper ideal, it follows that either \( J \subseteq H \) or \( J = E^0 \). Since \( s(\alpha) \in J \setminus H \), we must have \( J = E^0 \). Choose any element \( w \in H \). Since \( w \in J \) it follows that there exists \( v \in J \) with \( w \geq v \). But since \( w \geq v \) and \( H \) is hereditary, it follows that \( v \in H \). Hence \( v \in J \cap H \), and there is a path from \( s(\alpha) \) to a vertex in \( H \). Choose a path \( \mu = \mu_1 \mu_2 \ldots \mu_n \) with \( s(\mu) = s(\alpha) \), \( r(\mu_{n-1}) \notin H \), and \( r(\mu) \notin H \). Since \( r(\mu_{n-1}) \notin H \) the previous paragraph shows that there exists a path \( \nu \) with \( s(\nu) = r(\mu_{n-1}) \) and \( r(\nu) = s(\alpha) \). Let \( \gamma := \nu \mu_1 \ldots \mu_{n-1} \) and let \( f := \mu_n \). Then \( \gamma \) is a cycle in \( E \setminus H \) and \( f \) is an edge with \( s(f) = s(\gamma) \) and \( r(f) \in H \). Furthermore, since \( s(\alpha) \) is a vertex on the cycle \( \gamma \), we see that for any \( x \in E^0 \) we have \( x \geq s(\gamma) \) if and only if \( x \geq s(\alpha) \). It follows from the previous paragraph that if \( x \in E^0 \), then \( x \geq s(\gamma) \) if and only if \( x \in E^0 \setminus H \).

\[ \square \]

**Remark 3.6.** Note that the conclusion of the above lemma does not hold if \( I \) is a maximal proper ideal that is not a largest proper ideal. For example, if \( E \) is the graph

\[
\begin{array}{ccc}
  v & \xrightarrow{} & x \\
  \downarrow & & \downarrow \\
  w & \xrightarrow{} & x
\end{array}
\]

and \( H = \{ v \} \) then \( I := I_H \) is a maximal ideal that is AF, and \( C^*(E)/I_H \cong M_2(\mathcal{O}_2) \) is purely infinite. However, there is no edge from the base point of a cycle to \( H = \{ v \} \).
Lemma 3.9. Let equivalence of (ii) and (iii) is shown in [Hje01, Lemma 2.1].

Proof. I write orthogonal projections in $A$. Since $p, q \in A$, we have $p \sim q$ and $p \perp q$. Thus, $v \in A$ such that $p = vv^*$ and $v^*v \leq q$. Note that $p \preceq q$ and $q \preceq p$ does not imply that $p \sim q$ (unless $A$ is finite).

If $e \in G^1$ then we see that $p_{r(e)} = s_e^*s_e$ and $s_es_e^* \preceq p_{s(e)}$. Therefore $p_{r(e)} \preceq p_{s(e)}$. More generally we see that $v \geq w$ implies $p_w \preceq p_v$.

Lemma 3.8. Let $A$ be a $C^*$-algebra with an increasing countable approximate unit $\{p_n\}_{n=1}^\infty$ consisting of projections. Then the following are equivalent.

(i) $A$ is stable.

(ii) For every projection $p \in A$ there exists a projection $q \in A$ such that $p \sim q$ and $p \perp q$.

(iii) For all $n \in \mathbb{N}$ there exists $m > n$ such that $p_n \preceq p_m - p_n$

Proof. The equivalence of (i) and (ii) is shown in [Hir98, Theorem 3.3]. The equivalence of (ii) and (iii) is shown in [Hje01, Lemma 2.1].

Lemma 3.9. Let $A$ be a $C^*$-algebra. Suppose $p_1, p_2, \ldots, p_n$ are mutually orthogonal projections in $A$, and $q_1, q_2, \ldots, q_n$ are mutually orthogonal projections in $A$ with $p_i \sim q_i$ for $1 \leq i \leq n$. Then $\sum_{i=1}^n p_i \sim \sum_{i=1}^n q_i$.

Proof. Since $p_i \sim q_i$ there exists $v_i \in A$ such that $v_i^*v_i = p_i$ and $v_iv_i^* = q_i$. Thus for $i \neq j$ we have $v_i^*v_j = v_j^*v_i = v_i^*v_i = v_j^*v_j = 0$ and $v_iv_j = 0$. Hence $(\sum_{i=1}^n v_i)^*\sum_{i=1}^n v_i = \sum_{i=1}^n v_i^*v_i = \sum_{i=1}^n p_i$ and $\sum_{i=1}^n v_i(\sum_{i=1}^n v_i)^* = \sum_{i=1}^n v_iv_i^* = \sum_{i=1}^n q_i$. Thus $\sum_{i=1}^n p_i \sim \sum_{i=1}^n q_i$.

Proposition 3.10. Let $E$ be a graph with no breaking vertices, and suppose that $I$ is a largest proper ideal of $C^*(E)$ and such that $C^*(E)/I$ is purely infinite and $I$ is AF. Then there exists a projection $p \in C^*(E)$ such that $pC^*(E)p$ is a full corner of $C^*(E)$ and $pIp$ is stable.

Proof. Lemma 3.5 implies that $I = I_{(H, B_H)}$ for some saturated hereditary subset $H$ of $E^0$, and there exists a cycle $\gamma$ in $E \setminus H$ and an edge $f \in E^1$ with $s(f) = s(\gamma)$ and $r(f) \in H$; and furthermore, if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \setminus H$. Since $E$ has no breaking vertices, we have that $B_H = \emptyset$ so that $I_{(H, B_H)}$ is the ideal generated by $\{p_v : v \in H\}$ and we may write $I_{(H, B_H)}$ as $I_H$.

Let $v = s(f) = s(\gamma)$ and let $w = r(f)$. Define $p := p_v + p_w$. Suppose $J \triangleleft C^*(E)$ and $pC^*(E)p \subseteq J$. Since $v \notin H$ we see that $p_v \notin I$ and hence $p_v \in pC^*(E)p \cap I \subseteq J \setminus I$. Thus $J \nsubseteq I$ and the fact that $I$ is a largest proper ideal implies that $J = C^*(E)$. Hence $pC^*(E)p$ is a full corner of $C^*(E)$.

In addition, since there are no breaking vertices

$$pIp = pIHp$$
\[ s^* \alpha \beta = p \left( \text{span}\{ s_\alpha s_\beta^*: r(\alpha) = r(\beta) \in H \} \right) p \]
\[ = \text{span}\{ p s_\alpha s_\beta^* p : r(\alpha) = r(\beta) \in H \} \]
\[ = \text{span}\{ s_\alpha s_\beta^*: r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{ v, w \} \}. \]

Let \( S := \{ \alpha \in E^*: s(\alpha) = v \text{ and } r(\alpha) = w \} \). Since \( S \) is a countable set we may list the elements of \( S \) and write \( S = \{ \alpha_1, \alpha_2, \ldots \} \). Define \( p_0 := p_w \) and \( p_n := p_w + \sum_{k=1}^{\infty} s_{\alpha_k} s_{\alpha_k}^* \) for \( n \in \mathbb{N} \).

We will show that for \( \mu, \nu \in S \) we have
\[
(3.1) \quad s^*_\mu s^*_\nu := \begin{cases} p_{r(\mu)} & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}
\]

First suppose that \( s^*_\mu s^*_\nu \neq 0 \). Then one of \( \mu \) and \( \nu \) must extend the other. Suppose \( \mu \) extends \( \nu \). Then \( \mu = \nu \lambda \) for some \( \lambda \in E^* \). Thus \( s(\lambda) = r(\nu) = w \) and \( r(\lambda) = r(\mu) = w \). However, \( I_H \) is an AF-algebra, and \( C^*(E_H) \) is strongly Morita equivalent to \( I_H \) \cite{BHRS}, so \( C^*(E_H) \) is an AF-algebra. Thus \( E_H \) contains no cycles. Since \( \lambda \) is a path in \( E_H \) with \( s(\lambda) = r(\lambda) = w \), and since \( E_H \) contains no cycles, we may conclude that \( \lambda = w \). Thus \( \mu = \nu \). A similar argument works when \( \nu \) extends \( \mu \). Hence the equation in (3.1) holds. It follows that the elements of the set \( \{ s_\alpha s_\alpha^*: \alpha \in S \} \cup \{ p_w \} \) are mutually orthogonal projections, and hence \( \{ p_n \}_{n=0}^{\infty} \) is an approximate unit for \( pI_P \).

Next we shall show that \( \{ p_n \}_{n=0}^{\infty} \) is an approximate unit for \( pI_P \). Given \( s_\alpha s_\beta^* \) with \( r(\alpha) = r(\beta) \in H \) and \( s(\alpha), s(\beta) \in \{ v, w \} \), we consider two cases.

**Case I:** \( s(\alpha) = w \). Then for any \( \alpha_k \in S \) we see that \( s_{\alpha_k} s_{\alpha_k}^* s_\alpha s_\beta^* = s_{\alpha_k} s_{\alpha_k}^* p_w s_\alpha s_\beta^* = 0 \). In addition, \( p_w s_\alpha s_\beta^* = s_\alpha s_\beta^* \). Thus \( \lim_{n \to \infty} p_n s_\alpha s_\beta^* = s_\alpha s_\beta^* \).

**Case II:** \( s(\alpha) = v \). Then \( \alpha = \alpha_j \lambda \) for some \( \alpha_j \in S \) and some \( \lambda \in E_H^* \) with \( s(\lambda) = w \). We have \( p_{r(\alpha)} s_{\alpha \beta} = 0 \), and also (3.1) implies that
\[
(s_{\alpha_k} s_{\alpha_k}^*) s_\alpha s_\beta^* = s_{\alpha_k} s_{\alpha_k}^* s_{\alpha_j} s_\lambda s_\beta^* = \begin{cases} s_\alpha s_\beta^* & k = j \\ 0 & k \neq j \end{cases}
\]

Thus \( \lim_{n \to \infty} p_n s_{\alpha_k} s_{\alpha_k}^* = s_{\alpha_k} s_{\alpha_k}^* \).

The above two cases imply that \( \lim_{n \to \infty} p_n x = x \) for any \( x \in \text{span}\{ s_\alpha s_\beta^*: r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{ v, w \} \} \). Furthermore, an \( \epsilon/3 \)-argument shows that \( \lim_{n \to \infty} p_n x = x \) for any \( x \in pI_{HP} = \text{span}\{ s_\alpha s_\beta^*: r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{ v, w \} \} \). A similar argument shows that \( \lim_{n \to \infty} x p_n = x \) for any \( x \in pI_{HP} \). Thus \( \{ p_n \}_{n=1}^{\infty} \) is an approximate unit for \( pI_{HP} \).

We shall now show that \( pI_{HP} \) is stable. For each \( n \in \mathbb{N} \) define
\[
\lambda^n := \gamma \gamma \cdots \gamma f.
\]

For any \( k, n \in \mathbb{N} \) we have
\[
s_{\lambda^n} s_{\lambda^n}^* s_{\lambda^n} s_{\lambda^n} \sim s_{\lambda^n} s_{\lambda^n} = p_{r(\lambda^n)} = p_w = s_{\alpha_k} s_{\alpha_k} \sim s_{\alpha_k} s_{\alpha_k}^*.
\]
For any $n \in \mathbb{N}$ choose $q$ large enough that $|\lambda^q| \geq |\alpha_k|$ for all $1 \leq k \leq n$. Then for all $j \in \mathbb{N}$ we see that $\lambda^{q+j} \in S$ and $\lambda^{q+j} \neq \alpha_k$ for all $1 \leq k \leq n$. Thus for any $1 \leq k \leq n$ we have

$$s_{\alpha_k}^* s_{\alpha_k} \sim s_{\alpha_k}^* s_{\alpha_k} = p_r(\alpha_k) = p_w = p_r(\lambda^{q+k}) = s_{\lambda^{q+k}}^* s_{\lambda^{q+k}} \sim s_{\lambda^{q+k}}^* s_{\lambda^{q+k}}$$

and

$$p_w = s_{\lambda^{q+k}}^* \sim s_{\lambda^{q+k}}^*.$$ 

It follows from Lemma 3.9 that

$$p_n = p_w + \sum_{k=1}^{n} s_{\alpha_k}^* s_{\alpha_k} \lesssim \sum_{k=0}^{n} s_{\lambda^{q+k}}^* s_{\lambda^{q+k}} \prec p_m - p_n$$

where $m$ is chosen large enough that $\lambda^{q+k} \in \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ for all $0 \leq k \leq n$. Lemma 3.8 shows that $p I_{H\mathcal{P}}$ is stable. □

4. Classification

In this section we state and prove our main results. We apply the methods of [ERR] to classify certain extensions of graph $C^*$-algebras in terms of their six-term exact sequences of $K$-groups. To do this we will need to discuss classes of $C^*$-algebras satisfying various properties. We give definitions of these properties here, and obtain a lemma that is a consequence of [ERR, Theorem 3.10].

Definition 4.1 (see Definition 3.2 of [ERR]). We will be interested in classes $\mathcal{C}$ of separable nuclear unital simple $C^*$-algebras in the bootstrap category $\mathcal{N}$ satisfying the following properties:

(I) Any element of $\mathcal{C}$ is either purely infinite or stably finite.

(II) $\mathcal{C}$ is closed under tensoring with $M_n$, where $M_n$ is the $C^*$-algebra of $n$ by $n$ matrices over $\mathcal{C}$.

(III) If $A$ is in $\mathcal{C}$, then any unital hereditary $C^*$-subalgebra of $A$ is in $\mathcal{C}$.

(IV) For all $A$ and $B$ in $\mathcal{C}$ and for all $x$ in $KK(A,B)$ which induce an isomorphism from $(K_+^+(A), [1_A])$ to $(K_+^+(A), [1_B])$, there exists a *-isomorphism $\alpha : A \rightarrow B$ such that $KK(\alpha) = x$.

Definition 4.2. If $B$ is a separable stable $C^*$-algebra, then we say that $B$ has the corona factorization property if every full projection in $\mathcal{M}(B)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

Lemma 4.3 (Cf. Theorem 3.10 of [ERR]). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of unital nuclear separable simple $C^*$-algebras in the bootstrap category $\mathcal{N}$ satisfying the properties of Definition 4.1. Let $A_1$ and $A_2$ be in $\mathcal{C}_2$ and let $B_1$ and $B_2$ be in $\mathcal{C}_1$ with $B_1 \otimes \mathbb{K}$ and $B_2 \otimes \mathbb{K}$ satisfying the corona factorization property. Let

$$\varepsilon_1 : \begin{array}{ccc} 0 & \longrightarrow & B_1 \otimes \mathbb{K} & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
\end{array}$$

and

$$\varepsilon_2 : \begin{array}{ccc} 0 & \longrightarrow & B_2 \otimes \mathbb{K} & \longrightarrow & E_2 & \longrightarrow & A_2 & \longrightarrow & 0 \\
\end{array}$$

be essential and unital extensions. If $K_{size}(\varepsilon_1) \cong K_{size}(\varepsilon_2)$, then $E_1 \otimes \mathbb{K} \cong E_2 \otimes \mathbb{K}$. 

ON THE CLASSIFICATION OF NONSIMPLE GRAPH C*-ALGEBRAS 11
Proof. Tensoring the extension $\mathfrak{e}_1$ by $K$ we obtain a short exact sequence $\mathfrak{e}'_1$ and and vertical maps

$$
\mathfrak{e}_1 : \quad 0 \rightarrow B_1 \otimes K \rightarrow E_1 \rightarrow A_1 \rightarrow 0
$$

$$
\mathfrak{e}'_1 : \quad 0 \rightarrow (B_1 \otimes K) \otimes K \rightarrow E_1 \otimes K \rightarrow A_1 \otimes K \rightarrow 0
$$

from $\mathfrak{e}_1$ into $\mathfrak{e}'_1$ that are full inclusions. These full inclusions induce isomorphisms of $K$-groups and hence we have that $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}'_1)$. In addition, since $\mathfrak{e}_1$ is essential, $B_1 \otimes K$ is an essential ideal in $E_1$, and the Rieffel correspondence between the strongly Morita equivalent $C^*$-algebras $E_1$ and $E_1 \otimes K$ implies that $(B_1 \otimes K) \otimes K$ is an essential ideal in $E_1 \otimes K$, so that $\mathfrak{e}'_1$ is an essential extension. Furthermore, since $B_1 \otimes K$ is stable and $\mathfrak{e}_1$ is essential and full [ERR, Proposition 1.5], it follows from [ERR, Proposition 1.6] that $\mathfrak{e}'_1$ is full. Moreover, since $K \otimes K \cong K$, we may rewrite $\mathfrak{e}'_1$ as

$$
\mathfrak{e}'_1 : \quad 0 \rightarrow B_1 \otimes K \rightarrow E_1 \otimes K \rightarrow A_1 \otimes K \rightarrow 0.
$$

By a similar argument, there is an essential and full extension

$$
\mathfrak{e}_2' : \quad 0 \rightarrow B_2 \otimes K \rightarrow E_2 \otimes K \rightarrow A_2 \otimes K \rightarrow 0
$$

such that $K_{\text{six}}(\mathfrak{e}_2') \cong K_{\text{six}}(\mathfrak{e}_2)$. Thus $K_{\text{six}}(\mathfrak{e}_1') \cong K_{\text{six}}(\mathfrak{e}_2')$, and [ERR, Theorem 3.10] implies that $E_1 \otimes K \cong E_2 \otimes K$. \qed

Lemma 4.4. Let $A$ be a $C^*$-algebra and let $I$ be a largest proper ideal of $A$. If $p \in A$ is a full projection, then the inclusion map $pIp \hookrightarrow I$ and the inclusion map $pAp/pIp \hookrightarrow A/I$ are both full inclusions.

Proof. Since $p$ is a full projection, we see that $A$ is Morita equivalent to $pAp$ and the Rieffel correspondence between ideals takes the form $J \mapsto pJp$. If $J$ is an ideal of $I$ with $pIp \subseteq J$, then by compressing by $p$ we obtain $pIp \subseteq pJp$. Since the Rieffel correspondence is a bijection, this implies that $I \subseteq J$, and because $J$ is an ideal contained in $I$, we get that $I = J$. Hence $pIp \hookrightarrow I$ is a full inclusion. Furthermore, because $I$ is a largest proper ideal of $A$, we know that $A/I$ is simple and thus $pAp/pIp \hookrightarrow A/I$ is a full inclusion. \qed

Theorem 4.5. If $A$ is a graph $C^*$-algebra with exactly one proper nontrivial ideal $I$, then $A$ classified up to stable isomorphism by the six-term exact sequence

$$
K_0(I) \rightarrow K_0(A) \rightarrow K_0(A/I)
$$

$$
K_1(A/I) \leftarrow K_1(A) \leftarrow K_1(I)
$$

with all $K_0$-groups considered as ordered groups. In other words, if $A$ is a graph $C^*$-algebras with precisely one proper nontrivial ideal $I$, if $A'$ is a
graph $C^*$-algebras with precisely one proper nontrivial ideal $I'$, and if
\[
\varepsilon_1: \quad 0 \to I \to A \to A/I \to 0
\]
\[
\varepsilon_2: \quad 0 \to I' \to A' \to A'/I' \to 0
\]
are the associated extensions, then $A \otimes K \cong A' \otimes K$ if and only if $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$.

Moreover, in cases $[1\infty]$, $[\infty1]$, and $[\infty\infty]$, the order structure on $K_0(A)$ may be removed from the invariant leaving it still complete. And in case $[11]$, the ordered group $K_0(A)$ is a complete invariant in its own right.

**Proof.** It is straightforward to show that $A \otimes K \cong A' \otimes K$ implies that $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$. Thus we need only establish the converse. To do this, we begin by assuming that $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$.

We define $\mathcal{C}_{API}$ as the union of the class of unital simple and separable AF-algebras and the class of simple, nuclear, unital, and separable purely infinite $C^*$-algebras in the bootstrap category. The category $\mathcal{C}_{API}$ meets all the requirements in the list in Definition 4.1: We clearly have that each algebra in $\mathcal{C}_{API}$ is either purely infinite or stably finite, and that $\mathcal{C}_{API}$ is closed under passing to matrices and unital hereditary subalgebras. We also need to prove that the Elliott invariant is complete for $\mathcal{C}_{API}$, and this follows by the classification results of Elliott [Ell76, Theorem 4.3] and Kirchberg-Phillips (see [Kir, Theorem C] and [Phi00, §4.2]) after noting that the classes are obviously distinguishable by the nature of the positive cone in $K_0$. Finally, as recorded in [ERR, Theorem 3.9], the stabilizations of the $C^*$-algebras in $\mathcal{C}_{API}$ all have the corona factorization property according to [KN06, Theorem 5.2] and [Ng, Proposition 2.1].

It follows from [DHS03, Lemma 1.3] and [BHRS, Corollary 3.5] that $I$ and $A/I$ are simple graph $C^*$-algebras and thus each of $I$ and $A/I$ is either an AF-algebra or a purely infinite algebra [DT05, Remark 2.16]. Similarly for $I'$ and $A'/I'$. Since $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$, we see that $K_0(I) \cong K_0(I')$ and $K_0(A/I) \cong K_0(A'/I')$ as ordered groups. By considering the positive cone in these groups, we may conclude that $I$ and $I'$ are either both purely infinite or both AF-algebras, and also $A/I$ and $A'/I'$ are either both purely infinite or both AF-algebras. Thus $A$ and $A'$ both fall into one of the four cases described in Remark 3.2.

**Cases** $[\infty\infty]$, $[\infty1]$

Write $A = C^*(E)$ for some graph $E$. Since $I$ is a largest proper ideal, Lemma 3.3 implies that $I = I_{(H,B)}$ for some saturated hereditary subset $H \subseteq E^0$. If we let $v \in E^0 \setminus H$, and define $p := p_v$, then $p \notin I$. Since $I$ is a largest proper ideal in $A$, this implies that the projection $p$ is full. Thus we obtain a full hereditary subalgebra $pAp$, and as noted in Lemma 4.4 we
have that all vertical maps in
\[
\varepsilon'_1 : \begin{array}{c}
0 \\
\downarrow \varepsilon'_1 \\
I
\end{array} \longrightarrow \begin{array}{c}
pIp \\
\downarrow \varepsilon'_1 \\
A
\end{array} \longrightarrow \begin{array}{c}
pAp \\
\downarrow \varepsilon'_1 \\
A/I
\end{array} \longrightarrow 0
\]
\[
\varepsilon_1 : \begin{array}{c}
0 \\
\downarrow \varepsilon_1 \\
E
\end{array} \longrightarrow \begin{array}{c}
I \\
\downarrow \varepsilon_1 \\
A
\end{array} \longrightarrow \begin{array}{c}
A \\
\downarrow \varepsilon_1 \\
A/I
\end{array} \longrightarrow 0
\]
are full inclusions. It follows that all of the above maps induce isomorphisms on the $K$-groups and $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon'_1)$.

In addition, since $pIp$ is nonunital and purely infinite, the ideal $pIp$ is stable (by Zhang’s dichotomy) and we may write $pIp \cong B_1 \otimes K$ for a suitably chosen $B_1 \in C_{\text{AF}}$. We now let $E_1 := pAp$ and $A_1 := pAp/pIp$. With this notation, $\varepsilon'_1$ takes the form
\[
\varepsilon'_1 : \begin{array}{c}
0 \\
\downarrow \varepsilon'_1 \\
B_1 \otimes K
\end{array} \longrightarrow \begin{array}{c}
E_1 \\
\downarrow \varepsilon'_1 \\
A_1
\end{array} \longrightarrow 0
\]
with $B_1$ and $A_1$ unital $C^*$-algebras in $C_{\text{AF}}$. Furthermore, $\varepsilon'_1$ is an essential extension because the ideal $I$ is a largest proper ideal in $A$, and thus also the ideal $pIp \cong B_1 \otimes K$ is a largest proper ideal in $A_1 \otimes K$, which implies that $pIp \cong B_1 \otimes K$ is an essential ideal.

By a similar argument, we may find an extension
\[
\varepsilon'_2 : \begin{array}{c}
0 \\
\downarrow \varepsilon'_2 \\
B_2 \otimes K
\end{array} \longrightarrow \begin{array}{c}
E_2 \\
\downarrow \varepsilon'_2 \\
A_2
\end{array} \longrightarrow 0
\]
with $E_2 := qA'q$ for a full projection $q \in A'$, the $C^*$-algebras $B_2$ and $A_2$ in $C_{\text{AF}}$ with $B_2 \otimes K$ satisfying the corona factorization property, and $\varepsilon'_2$ an essential and full extension with $K_{\text{six}}(\varepsilon'_2) \cong K_{\text{six}}(\varepsilon_2)$. It follows from Lemma 4.3 that $E_1 \otimes K \cong E_2 \otimes K$, or equivalently, that $pAp \otimes K \cong qA'q \otimes K$. Furthermore, because $pAp$ is a full corner of $A$, and $qAq$ is a full corner of $A'$, we obtain that $pAp \otimes K \cong A \otimes K$ and $qA'q \otimes K \cong A' \otimes K$. It follows that $A \otimes K \cong A' \otimes K$. We also observe that in this case the order structure on $K_0(A)$ is a redundant part of the invariant.

**Case $[1,\infty]$**

As seen in Lemma 3.1, we may write $A = C^*(E)$ where $E$ has no breaking vertices. By Proposition 3.10 there exists a projection $p \in A$ such that $pAp$ is a full corner inside $A$, and $pIp$ is stable. Moreover, since $I$ is an AF-algebra by hypothesis and $pIp$ is a hereditary subalgebra of $I$, it follows from [D176b] Theorem 3.1 that $pIp$ is an AF-algebra. Hence we may choose a unital AF-algebra $B_1$ with $pIp \cong B_1 \otimes K$. The extension
\[
\varepsilon'_1 : \begin{array}{c}
0 \\
\downarrow \varepsilon'_1 \\
B_2 \otimes K
\end{array} \longrightarrow \begin{array}{c}
pAp \\
\downarrow \varepsilon'_1 \\
pAp/pIp
\end{array} \longrightarrow 0
\]
is essential. In addition, an argument as in Cases $[\infty,\infty]$, $[\infty,1]$ shows that $K_{\text{six}}(\varepsilon'_1) \cong K_{\text{six}}(\varepsilon_1)$. We may perform a similar argument for $A'$, and arguing as in Cases $[\infty,\infty]$, $[\infty,1]$ and applying Lemma 4.3 we obtain that $A \otimes K \cong A' \otimes K$. Again, the order structure on $K_0(A)$ is a redundant part of the invariant.
Case [11]
Since $I$ and $A/I$ are AF-algebras, it follows from a result of Brown that $A$ is an AF-algebra \cite{Bro81} (or see \cite{Eff81} §9.9 for a detailed proof). Similarly, $A'$ is an AF-algebra. It follows from Elliott’s Theorem that $K_0(A)$ order isomorphic to $K_0(A')$ implies that $A \otimes K \cong A' \otimes K$. Moreover, in this case the ordered group $K_0(A)$ is a complete invariant. □

Remark 4.6. Joint work in progress by Ruiz and the first named author provides information about the necessity of using order on the $K_0$-groups of $K_{\text{six}}(-)$. There are examples of pairs of non-isomorphic stable AF-algebras $A$ and $A'$ with exactly one ideal $I$ and $I'$ such that $K_{\text{six}}(e) \cong K_{\text{six}}(e')$ with group isomorphisms which are positive at $K_0(I)$ and $K_0(A/I)$ but not at $K_0(A)$. By \cite{KST}, Corollary 4.8, $A$ and $A'$ may be realized as graph $C^*$-algebras. In the other cases, one may prove that any isomorphism between $K_{\text{six}}(e)$ and $K_{\text{six}}(e')$ will automatically be positive at $K_0(A)$ if it is positive at $K_0(I)$ and $K_0(A/I)$. Thus it is possible that any isomorphism of our reduced invariant lifts to a $*$-isomorphism in the $[\infty \infty]$, $[\infty 1]$, and $[1 \infty]$ cases, but this has only been established in the $[\infty \infty]$ case, cf. \cite{ER}.

Our proof of Theorem 4.5 can be modified slightly to give us an additional result.

**Theorem 4.7.** If $A$ is a the $C^*$-algebra of a graph satisfying Condition (K), and if $A$ has a largest proper ideal $I$ such that $I$ is an AF-algebra, then $A$ is classified up to stable isomorphism by the six-term exact sequence

$$
\begin{array}{cccccc}
K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) & \\
\downarrow & & \downarrow & & \downarrow & \\
K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I)
\end{array}
$$

with $K_0(I)$ considered as an ordered group.

In other words, if $A$ is the $C^*$-algebra of a graph satisfying Condition (K) with a largest proper ideal $I$ that is an AF-algebra, if $A'$ is the $C^*$-algebra of a graph satisfying Condition (K) with a largest proper ideal $I'$ that is an AF-algebra, and if

\begin{align*}
\varepsilon_1 : & \quad 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \\
\varepsilon_2 : & \quad 0 \longrightarrow I' \longrightarrow A' \longrightarrow A'/I' \longrightarrow 0
\end{align*}

are the associated extensions, then $A \otimes K \cong A' \otimes K$ if and only if $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$.

**Proof.** To begin, using the desingularization of \cite{DT05} we may find a row-finite graph $F$ such that $C^*(F)$ is stably isomorphic to $A$. Since $C^*(F)$ is Morita equivalent to $A$, the $C^*$-algebra $C^*(F)$ has a largest proper ideal that is an AF-algebra, and the associated six-term exact sequence of $K$-groups is isomorphic to $K_{\text{six}}(\varepsilon_1)$. Hence we may replace $A$ by $C^*(F)$ for
the purposes of the proof. Likewise for $A'$. Thus we may, without loss of
generality, assume that $A$ and $A'$ are $C^*$-algebras of row-finite graphs, and
in particular that $A$ and $A'$ are $C^*$-algebras of graphs with no breaking
vertices. To obtain the result, we simply argue as in Case $1\infty$ of the proof
of Theorem 4.5 using [ERR, Theorem 3.13] in place of [ERR, Theorem 3.10],
and noting that Proposition 3.10 applies since the graphs have no breaking
vertices. \hfill \Box

5. Examples

To illustrate our methods we give a complete classification, up to stable
isomorphism, of all $C^*$-algebras of graphs with two vertices that have pre-
cisely one proper nontrivial ideal. Combined with other results, this allows
us to give a complete classification of all $C^*$-algebras of graphs satisfying
Condition (K) with exactly two vertices.

If $E$ is a graph with two vertices, and if $C^*(E)$ has exactly one proper
ideal, then $E$ must have exactly one proper nonempty saturated hereditary
subset with no breaking vertices. This occurs precisely when the vertex
matrix of $E$ has the form

$$
\begin{bmatrix}
a & b \\
0 & d
\end{bmatrix}
$$

where $a, d \in \{0, 2, 3, \ldots, \infty\}$ and $b \in \{1, 2, 3, \ldots, \infty\}$ with the extra condi-
tions

$$
a = 0 \implies b = \infty \quad \text{and} \quad b = \infty \implies (a = 0 \text{ or } a = \infty),
$$

Computing $K$-groups using [DT02], we see that in all of these cases the
$K_1$-groups of $C^*(E)$, the unique proper nontrivial ideal $I$, and the quo-
tient $C^*(E)/I$ all vanish. Thus the six-term exact sequence becomes $0 \to K_0(I) \to K_0(C^*(E)) \to K_0(C^*(E)/I) \to 0$, and using [DT02] to compute
the $K_0$-groups and the induced maps we obtain the following cases.
Proof. Suppose $C^*(E) \otimes K \cong C^*(E') \otimes K$. Then $K_0(I) \cong K_0(I')$ as ordered groups and $K_0(C^*(E)/I) \cong K_0(C^*(E')/I')$ as ordered groups. From a consideration of the invariants in the above table, this implies that $a = a'$, $d = d'$, and the invariants for $C^*(E)$ and $C^*(E')$ both fall into the same case (i.e. the same row) of the table. Thus we need only consider the two cases described in (3)(a) and (3)(b).

**Case 1**: $a \in \{2, \ldots \}$ and $d \in \{0, \infty \}$. 

where “$n$” indicates an integer $\geq 2$, “$\mathbb{Z}_{++}$” indicates a copy of $\mathbb{Z}$ ordered with $\mathbb{Z}_{+} = \mathbb{N}$ and “$\mathbb{Z}_+$” indicates a copy of $\mathbb{Z}$ ordered with $\mathbb{Z}_{+} = \mathbb{Z}$. In addition, in all cases we have written the middle group in such a way that the map from $K_0(I)$ to $K_0(C^*(E))$ is $[x] \mapsto [(x, 0)]$, and the map from $K_0(C^*(E))$ to $K_0(C^*(E)/I)$ is $[(x, y)] \mapsto [y]$. Note that in all but the first case, the order structure of the middle $K_0$-groups is irrelevant and need not be computed.

**Theorem 5.1.** Let $E$ and $E'$ be graphs each with two vertices such that $C^*(E)$ and $C^*(E')$ each have exactly one proper nontrivial ideal, and write the vertex matrix of $E$ as $[a \ b]$, and the vertex matrix of $E'$ as $[a' \ b']$. Then

$$C^*(E) \otimes K \cong C^*(E') \otimes K$$

if and only if the following three conditions hold:

1. $a = a'$
2. $d = d'$
3. If $a \in \{2, \ldots \}$ then
   - (a) If $d \in \{0, \infty \}$ then $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}_{a-1}$
   - (b) If $d \in \{2, \ldots \}$ then $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1,d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$.

Proof. Suppose $C^*(E) \otimes K \cong C^*(E') \otimes K$. Then $K_0(I) \cong K_0(I')$ as ordered groups and $K_0(C^*(E)/I) \cong K_0(C^*(E')/I')$ as ordered groups. From a consideration of the invariants in the above table, this implies that $a = a'$, $d = d'$, and the invariants for $C^*(E)$ and $C^*(E')$ both fall into the same case (i.e. the same row) of the table. Thus we need only consider the two cases described in (3)(a) and (3)(b).
In this case there are isomorphisms $\alpha$, $\beta$, and $\gamma$ such that
\[
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow \text{coker}(\begin{bmatrix} b \\ a-1 \end{bmatrix}) \rightarrow \mathbb{Z}_{a-1} \rightarrow 0 \\
\alpha \downarrow \beta \downarrow \gamma \\
0 \rightarrow \mathbb{Z} \rightarrow \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix}) \rightarrow \mathbb{Z}_{a-1} \rightarrow 0
\end{array}
\]
commutes. Since the only automorphisms on $\mathbb{Z}$ are $\pm \text{Id}$, we have that $\alpha(x) = \pm x$. Also, since the only automorphisms on $\mathbb{Z}_{a-1}$ are multiplication by a unit, $\gamma([x]) = [z][x]$ for some unit $[z] \in \mathbb{Z}_{a-1}$. By the commutativity of the left square $\beta([1,0]) = ([\pm 1,0])$. Also, by the commutativity of the right square, $\beta([0,1]) = ([y,z])$ for some $y \in \mathbb{Z}$. It follows from the $\mathbb{Z}$-linearity of $\beta$ that $\beta([r,s]) = ([\pm r + sy, sz])$, so $\beta$ is equal to left multiplication by the matrix $\begin{bmatrix} \pm 1 & y \\ 0 & z \end{bmatrix}$. We must have $\beta((b,a-1)) = ([0,0])$, and thus $\begin{bmatrix} (\pm b + (a-1)y, (a-1)z) \end{bmatrix} = ([0,0])$ in $\text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix})$. Hence $\pm b + (a-1)y = b't$ and $(a-1)z = (a-1)t$ for some $t \in \mathbb{Z}$. It follows that $z = t$ and $\pm b + (a-1)y = b'z$, so $b \equiv \pm z \text{ mod } (a-1)$. Since $\pm z$ is a unit for $\mathbb{Z}_{a-1}$ it follows that $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}_{a-1}$. Thus the condition in (a) holds.

**Case II:** $a \in \{2, \ldots \}$ and $d \in \{2, \ldots \}$. In this case there are isomorphisms $\alpha$, $\beta$, and $\gamma$ such that
\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}_{d-1} \rightarrow \text{coker}(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}) \rightarrow \mathbb{Z}_{a-1} \rightarrow 0 \\
\alpha \downarrow \beta \downarrow \gamma \\
0 \rightarrow \mathbb{Z}_{d-1} \rightarrow \text{coker}(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}) \rightarrow \mathbb{Z}_{a-1} \rightarrow 0
\end{array}
\]
commutes. Since the only automorphisms on $\mathbb{Z}_{d-1}$ are multiplication by a unit, we have that $\alpha([x]) = [z_1][x]$ for some unit $[z_1] \in \mathbb{Z}_{d-1}$. Likewise, $\gamma([x]) = [z_2][x]$ for some unit $[z_2] \in \mathbb{Z}_{a-1}$. By the commutativity of the left square $\beta([1,0]) = ([z_1,0])$. Also, by the commutativity of the right square, $\beta([0,1]) = ([y,z_2])$ for some $y \in \mathbb{Z}$. It follows from the $\mathbb{Z}$-linearity of $\beta$ that $\beta([r,s]) = ([z_1 r + ys, z_2 s])$, so $\beta$ is equal to left multiplication by the matrix $\begin{bmatrix} z_1 & y \\ 0 & z_2 \end{bmatrix}$. Since $\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ a-1 \end{bmatrix}$, we must have $\beta((b,a-1)) = ([0,0])$, and thus $\begin{bmatrix} (z_1 b + y(a-1), z_2 (a-1)) \end{bmatrix} = ([0,0])$ in $\text{coker}(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix})$. Hence $z_1 b + y(a-1) = (d-1)s + b't$ and $z_2 (a-1) = (a-1)t$ for some $s,t \in \mathbb{Z}$. It follows that $z_2 = t$ and $z_1 b + y(a-1) = (d-1)s + b'z_2$. Writing $(d-1)s - y(a-1) = k \text{gcd}(a-1,d-1)$ we obtain $z_1 b - z_2 b' = k \text{gcd}(a-1,d-1)$ so that $z_1 b \equiv z_2 b' \text{ mod } \text{gcd}(a-1,d-1)$ and $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\text{gcd}(a-1,d-1)}$. Thus the condition in (b) holds.

For the converse, we assume that the conditions in (1)–(3) hold. Consider the following three cases.
CASE I: $a = 0$ or $a = \infty$. In this case, by considering the invariants listed in the above table, we see that we may use the identity maps for the three vertical isomorphisms to obtain a commutative diagram. Thus the six-term exact sequences are isomorphic, and it follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

CASE II: $a \in \{2, \ldots\}$ and $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}_{a-1}$. Then $b \cong zb' \mod (a-1)$. Hence $zb' - b = (a-1)y$ for some $y \in \mathbb{Z}$. Consider $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\text{im} \begin{bmatrix} b \\ a-1 \end{bmatrix} \to \text{im} \begin{bmatrix} b' \\ a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta : \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix}) \to \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix})$. In addition, if we let $\alpha = \text{Id}$ and let $\gamma$ be multiplication by $[z]$, then it is straightforward to verify that the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} b \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\
\alpha & & \downarrow \beta & & \gamma \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0
\end{array}
\]

commutes. Since $\alpha$ and $\gamma$ are isomorphisms, an application of the five lemma implies that $\beta$ is an isomorphism. It follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

CASE III: Suppose that $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1,d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$. Then $z_1b - z_2b' = k \gcd(a-1, d-1)$ for some $k \in \mathbb{Z}$. Furthermore, we may write $k \gcd(a-1, d-1) = s(d-1) - y(a-1)$ for some $s, y \in \mathbb{Z}$. Consider $\begin{bmatrix} z_1 \\ 0 \\ z_2 \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\text{im} \begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix} \to \text{im} \begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta : \text{coker}(\begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix}) \to \text{coker}(\begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix}) \to \text{coker}(\begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix})$. In addition, if we let $\alpha$ be multiplication by $[z_1]$ and let $\gamma$ be multiplication by $[z_2]$, then it is straightforward to verify that the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\
\alpha & & \downarrow \beta & & \gamma \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} d-1 \\ 0 \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0
\end{array}
\]

commutes. Since $\alpha$ and $\gamma$ are isomorphisms, an application of the five lemma implies that $\beta$ is an isomorphism. It follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

Example 5.2. Consider the three graphs

[Graphs shown here]

\[\square\]
which all have graph $C^*$-algebras with precisely one proper nontrivial ideal. By Theorem 5.1, the $C^*$-algebras of the two first graphs are stably isomorphic to each other, but not to the $C^*$-algebra of the third graph.

**Remark 5.3.** We mention that with existing technology it would be very difficult to see directly that the $C^*$-algebras of the two first graphs in Example 5.2 are stably isomorphic. One approach would be to form the stabilized graphs (see [Tom04, §4]) and then attempt to transform one graph to the other through operations that preserve the stable isomorphism class of the associated $C^*$-algebra (e.g., in/outsplittings, delays). However, even in this concrete example it is unclear what sequence of operations would accomplish this and we speculate that it would not be possible using the types of operations mentioned above and their inverses. In addition, the second author’s Ph.D. thesis (see [Tom02] and the resulting papers [RTW], [Tom01], and [Tom03]) deals with extensions of graph $C^*$-algebras and shows that under certain circumstances two essential one-sink extensions of a fixed graph $G$ have stably isomorphic $C^*$-algebras if they determine the same class in $\text{Ext} C^*(G)$ ([Tom01] Theorem 4.1). In Example 5.2, the three displayed graphs are all essential one-sink extensions of the graph with one vertex and four edges, whose $C^*$-algebra is $\mathcal{O}_4$. We also have that $\text{Ext} \mathcal{O}_4 \cong \mathbb{Z}_3$, and the first two graphs in Example 5.2 determine the classes $[1]$ and $[2]$ in $\mathbb{Z}_3$, respectively. Consequently, we cannot apply [Tom01] Theorem 4.1, and we see that the methods of this paper have applications to situations not covered by [Tom01] Theorem 4.1. (As an aside, we mention that the second author has conjectured that if $G$ is a finite graph with no sinks or sources, if $C^*(G)$ is simple, and if $E_1$ and $E_2$ are one-sink extensions of $G$, then $C^*(E_1)$ is stably isomorphic to $C^*(E_2)$ if and only if there exists an automorphism on $\text{Ext} C^*(G)$ taking the class of the extension determined by $E_1$ to the class of the extension determined by $E_2$. We see that Example 5.2 is consistent with this conjecture since there is an automorphism of $\mathbb{Z}_3$ taking $[1]$ to $[2]$.)

Using the Kirchberg-Phillips Classification Theorem and our results in Theorem 5.1, we are able to give a complete classification of the stable isomorphism classes of $C^*$-algebras of graphs satisfying Condition (K) with exactly two vertices. We state this result in the following theorem. As one can see, there are a variety of cases and possible ideal structures for these stable isomorphism classes.

**Theorem 5.4.** Let $E$ and $E'$ be graphs satisfying Condition (K) that each have exactly two vertices. Let $A_E$ and $A_{E'}$ be the vertex matrices of $E$ and $E'$, respectively, and order the vertices of each so that $c \leq b$ and $c' \leq b'$. Then $C^*(E) \otimes K \cong C^*(E') \otimes K$ if and only if one of the following five cases occurs.
(i) \( A_E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( A_E = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \) with

\( b \neq 0 \) and \( c \neq 0 \) or \( (a = 0, 0 < b < \infty, c = 0, \text{ and } d \geq 2) \)

and

\( b' \neq 0 \) and \( c' \neq 0 \) or \( (a' = 0, 0 < b' < \infty, c' = 0, \text{ and } d' \geq 2) \)

and if \( B_E \) is the \( E^0 \times E^0_{\text{reg}} \) submatrix of \( A_E^t - I \) and \( B_{E'} \) is the \( (E')^0 \times (E')^0_{\text{reg}} \) submatrix of \( A_{E'}^t - I \), then

\[
\text{coker}(B_E : \mathbb{Z}^{E^0_{\text{reg}}} \to \mathbb{Z}^{E^0}) \cong \text{coker}(B_{E'} : \mathbb{Z}^{(E')^0_{\text{reg}}} \to \mathbb{Z}^{(E')^0})
\]

and

\[
\text{ker}(B_E : \mathbb{Z}^{E^0_{\text{reg}}} \to \mathbb{Z}^{E^0}) \cong \text{ker}(B_{E'} : \mathbb{Z}^{(E')^0_{\text{reg}}} \to \mathbb{Z}^{(E')^0}).
\]

In this case \( C^*(E) \) and \( C^*(E') \) are purely infinite and simple.

(ii) \( A_E = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \) and \( A_E = \begin{bmatrix} 0 & b' \\ 0 & 0 \end{bmatrix} \) with \( 0 < b < \infty \) and \( 0 < b' < \infty \). In this case \( C^*(E) \cong M_{b+1}(\mathbb{C}) \) and \( C^*(E') \cong M_{b'+1}(\mathbb{C}) \), so that both \( C^* \)-algebras are simple and finite-dimensional.

(iii) \( A_E = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \) and \( A_E = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \) with \( b \neq 0 \) and \( b' \neq 0 \),

\( a = 0 \implies b = \infty \) and \( b = \infty \implies (a = 0 \text{ or } a = \infty) \),

and

\( a' = 0 \implies b' = \infty \) and \( b' = \infty \implies (a' = 0 \text{ or } a' = \infty) \),

and the conditions (1)–(3) of Theorem 5.1 hold. In this case \( C^*(E) \) and \( C^*(E') \) each have exactly one proper nontrivial ideal and have ideal structure of the form

\[
\begin{vmatrix}
A \\
I \\
\{0\}
\end{vmatrix}
\]

(iv) \( A_E = \begin{bmatrix} a^\infty \\ 0 \\ d \end{bmatrix} \) and \( A_E = \begin{bmatrix} a'\infty \\ 0 \\ d' \end{bmatrix} \) with \( a \in \{2, 3, \ldots\} \) and \( a' \in \{2, 3, \ldots\} \),

and with \( a = a' \) and \( d = d' \). In this case \( C^*(E) \) and \( C^*(E') \) each have exactly two proper nontrivial ideals and have ideal structure of
the form
\[
\begin{array}{c}
A \\
I \\
J \\
\{0\}.
\end{array}
\]

(v) \(A_E = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\) and \(A_{E'} = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix}\) with
\((a = a' \text{ and } d = d')\) or \((a = d' \text{ and } d = a')\).

In this case \(C^*(E) \cong C^*(E') \cong I \oplus J\), where
\[
I := \begin{cases} \mathbb{O}_a & \text{if } a \geq 2 \\ \mathbb{C} & \text{if } a = 0 \end{cases}
\]
and
\[
J := \begin{cases} \mathbb{O}_d & \text{if } d \geq 2 \\ \mathbb{C} & \text{if } d = 0 \end{cases},
\]
and each \(C^*\)-algebra has exactly two proper nontrivial ideals and ideal structure of the form

\[
\begin{array}{c}
A \\
I \\
J \\
\{0\}.
\end{array}
\]

Remark 5.5. We are not able to classify \(C^*\)-algebras of graphs with exactly two vertices that do not satisfy Condition (K). For example if \(E\) and \(E'\) are graphs with vertex matrices \(A_E = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\) and \(A_{E'} = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}\), then \(C^*(E)\) and \(C^*(E')\) each have uncountably many ideals, and are extensions of \(C(\mathbb{T})\) by \(C(\mathbb{T} \otimes \mathbb{K})\). Using existing techniques, it is unclear when \(C^*(E)\) and \(C^*(E')\) will be stably isomorphic.

We conclude this section with an example showing an application of Theorem 4.7 to \(C^*\)-algebras with multiple proper ideals.

Example 5.6. Consider the two graphs
The ideal $I := I_{\{v, w\}}$ in $C^*(E)$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \to I \to C^*(E) \to C^*(E)/I \to 0$$

is

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \text{coker} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \to 0$$

where the middle map is $[(x, y)] \mapsto [(x, y, 0)]$. Likewise, the ideal $I' := I_{\{v', w'\}}$ in $C^*(E')$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \to I' \to C^*(E') \to C^*(E')/I' \to 0$$

is

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \text{coker} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \to 0$$

where the middle map is $[(x, y)] \mapsto [(x, y, 0)]$. If we define $\beta : \text{coker} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \to \text{coker} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ by $\beta [(x, y, z)] = [(x + z, y + z, z)]$, then we see that the diagram

$$\begin{array}{ccc}
0 & \to & \mathbb{Z} \oplus \mathbb{Z} \\
\text{id} & \downarrow & \beta \\
0 & \to & \text{coker} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\end{array}$$

commutes. An application of the five lemma shows that $\beta$ is an isomorphism. It follows from Theorem 4.7 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

In the examples above, both connecting maps in the six-term exact sequences vanish. Since all $C^*$-algebras considered (and, more generally, all graph $C^*$-algebras satisfying Condition (K)) have real rank zero, the exponential map $\partial : K_0(A/I) \to K_1(I)$ is always zero. However, the index map $\partial : K_1(A/I) \to K_0(I)$ does not necessarily vanish and may carry important information. In forthcoming work, the authors and Carlsen explain how to compute this map for graph $C^*$-algebras.

### 6. Stability of ideals

In this section we prove that if $A$ is a graph $C^*$-algebra that is not an AF-algebra, and if $A$ contains a unique proper nontrivial ideal $I$, then $I$ is stable.

**Definition 6.1.** If $v$ is a vertex in a graph $E$ we define

$$L(v) := \{ w \in E^0 : \text{ there is a path from } w \text{ to } v \}.$$ 

We say that $v$ is left infinite if $L(v)$ contains infinitely many elements.

**Definition 6.2.** If $E = (E^0, E^1, r, s)$ is a graph, then a graph trace on $E$ is a function $g : E^0 \to [0, \infty)$ with the following two properties:
(1) For any \( v \in G^0 \) with \( 0 < |s^{-1}(v)| < \infty \) we have \( g(v) = \sum_{s(e)=v} g(r(e)). \)
(2) For any infinite emitter \( v \in G^0 \) and any finite set of edges \( e_1, \ldots, e_n \in s^{-1}(v) \) we have \( g(v) \geq \sum_{i=1}^{n} g(r(e_i)). \)

We define the norm of a graph trace \( g \) to be the (possibly infinite) quantity 
\[ \|g\| := \sum_{v \in E^0} g(v), \]
and we say a graph trace \( g \) is bounded if \( \|g\| < \infty. \)

**Lemma 6.3.** Let \( E \) be a graph such that \( C^*(E) \) is simple. If there exists \( v \in E^0 \) such that \( v \) is left infinite, then \( C^*(E) \) is stable.

**Proof.** Since \( C^*(E) \) is simple, it follows from [DT05 Corollary 2.15] that \( E \) is cofinal. Therefore, the vertex \( v \) can reach every cycle in \( E \), and any vertex that is on a cycle in \( E \) is left infinite. In addition, if \( g : E^0 \to [0, \infty) \) is a bounded graph trace on \( E \), then since \( v \) is left infinite, it follows that \( g(v) = 0 \). Furthermore, it follows from [Tom06 Lemma 3.7] that 
\[ H := \{ w \in E^0 : g(w) = 0 \} \]
is a saturated hereditary subset of vertices. Since \( C^*(E) \) is simple, it follows from [DT05 Theorem 3.5] that the only saturated hereditary subsets of \( E \) are \( E^0 \) and \( \emptyset \). Because \( v \in H \), we have that \( H \neq \emptyset \) and hence \( H = E^0 \), which implies that \( g \equiv 0 \). Since we have shown that every vertex on a cycle in \( E \) is left infinite, and that there are no nonzero bounded graph traces on \( E \), it follows from [Tom06 Theorem 3.2(d)] that \( C^*(E) \) is stable. \( \square \)

**Proposition 6.4.** Let \( E \) be a graph such that \( C^*(E) \) contains a unique proper nontrivial ideal \( I \), and let \( \{ E^0, H, \emptyset \} \) be the saturated hereditary subsets of \( E \). Then there are two possibilities:

(1) The ideal \( I \) is stable; or
(2) The graph \( C^* \)-algebra \( C^*(E) \) is a nonunital AF-algebra, and \( H \) is infinite.

**Proof.** By Lemma 3.1 we see that \( E \) contains a unique saturated hereditary subset \( H \) not equal to either \( E^0 \) or \( \emptyset \), and also \( I = I_H \). In addition, it follows from [DHS03 Lemma 1.6] that \( I_H \) is isomorphic to the graph \( C^* \)-algebra \( C^*(H_E \emptyset) \), where \( H_E \emptyset \) is the graph described in [DHS03 Definition 1.4]. In particular, if we let
\[ F_H := \{ \alpha \in E^* : s(\alpha) \notin H, r(\alpha) \in H, \text{ and } r(\alpha_i) \notin H \text{ for } i < |\alpha| \} \]
then
\[ H_E \emptyset := H \cup F_H \quad \text{and} \quad H_E^1 := \{ e \in E^1 : s(e) \in H \} \cup \{ \overline{\alpha} : \alpha \in F_H \} \]
where \( s(\overline{\alpha}) = \alpha, r(\overline{\alpha}) = r(\alpha), \) and the range and source of the other edges is the same as in \( E \). Note that since \( I \) is the unique proper nontrivial ideal in \( C^*(E) \), we have that \( I \cong C^*(H_E \emptyset) \) is simple.

Consider three cases.

**Case I:** \( H \) is finite.

Choose a vertex \( v \in E^0 \setminus H \). By Lemma 3.1 \( v \) is not a sink in \( E \), and thus there exists an edge \( e_1 \in E^1 \) with \( s(e_1) = v \) and \( r(e_1) \notin H \). Continuing
inductively, we may produce an infinite path $e_1e_2e_3\ldots$ with $r(e_i) \notin H$ for all $i$. (Note that the vertices of this infinite path need not be distinct.) We shall show that for each $i$ there is a path from $r(e_i)$ to a vertex in $H$. Fix $i$, and let

$$X := \{ w \in E^0 : \text{there is a path from } r(e_i) \text{ to } w \}.$$

Then $X$ is a nonempty hereditary subset, and by Lemma 3.1 it follows that $X \cap H \neq \emptyset$. Thus there is a path from $r(e_i)$ to a vertex in $H$. Since this is true for all $i$, it must be the case that $F_H$ is infinite. In the graph $HE_0$ there is an edge from each element of $F_H$ to an element in $H$. Since $H$ is finite, this implies that there is a vertex in $H \subseteq HE_0$ that is reached by infinitely many vertices, and hence is left infinite. It follows from Lemma 6.4 that $I \cong C^*(HE_0)$ is stable. Thus we are in the situation described in (1).

**Case II:** $H$ is infinite, and $E$ contains a cycle.

Let $\alpha = \alpha_1 \ldots \alpha_n$ be a cycle in $E$. Since $H$ is hereditary, the vertices of $\alpha$ must either all lie outside of $H$ or all lie inside of $H$. If the vertices all lie in $H$, then the graph $HE_0$ contains a cycle, and since $C^*(HE_0)$ is simple, the dichotomy for simple graph $C^*$-algebras [DT05, Remark 2.16] implies that $C^*(HE_0)$ is purely infinite. Since $H$ is infinite, it follows that $HE_0$ is infinite and $C^*(HE_0)$ is nonunital. Because $C^*(HE_0)$ is a simple, separable, purely infinite, and nonunital $C^*$-algebra, Zhang’s Theorem [Zha90] implies that $I \cong C^*(HE_0)$ is stable. Thus we are in the situation described in (1).

If the vertices of $\alpha$ all lie outside $H$, then the set

$$X := \{ w \in E^0 : \text{there is a path from } r(\alpha_n) \text{ to } w \}$$

is a nonempty hereditary set. It follows from Lemma 3.1 that $X \cap H \neq \emptyset$. Thus there exists a vertex $v \in H$ and a path $\beta$ from $r(\alpha_n)$ to $v$ with $r(\beta_i) \notin H$ for $i < |\beta|$. Consequently there are infinitely many paths in $F_H$ that end at $v$ (viz. $\beta, \alpha\beta, \alpha\alpha\beta, \alpha\alpha\alpha\beta, \ldots$). Hence there are infinitely many vertices in $HE_0$ that can reach $v$, and $v$ is a left infinite vertex in $HE_0$. It follows from Lemma 6.3 that $I \cong C^*(HE_0)$ is stable. Thus we are in the situation described in (1).

**Case III:** $H$ is infinite, and $E$ does not contain a cycle.

Since $E$ does not contain a cycle, it follows from [DT05, Corollary 2.13] that $C^*(E)$ is an AF-algebra. In addition, since $H$ is infinite it follows that $E^0$ is infinite and $C^*(E)$ is nonunital. Thus we are in the situation described in (2). \(\square\)

**Corollary 6.5.** If $E$ is a graph with a finite number of vertices and such that $C^*(E)$ contains a unique proper nontrivial ideal $I$, then $I$ is stable. Furthermore, if $\{E^0, H, \emptyset\}$ are the saturated hereditary subsets of $E$, then $C^*(EH)$ is a unital $C^*$-algebra and $I \cong C^*(EH) \otimes K$.

**Proof.** Since $E^0$ is finite it is the case that $C^*(E)$ is unital, and it follows from Proposition 6.4 that $I$ is stable. Furthermore, since $I = IH$ it follows from [BPRS00, Theorem 4.1] and [BPRS00, Proposition 3.4] that $I$ is Morita equivalent to $C^*(EH)$. Since $I$ and $C^*(EH)$ are separable, it follows that $I$
and $C^*(E_H)$ are stably isomorphic. Thus $I \cong I \otimes K \cong C^*(E_H) \otimes K$. Finally, since $E_H = H \subseteq E^0$ is finite, $C^*(E_H)$ is unital. □

References


[Rae05] I. Raeburn, Graph algebras. CBMS Regional Conference Series in Mathematics, 103, Published for the Conference Board of the Mathematical Sciences,
ON THE CLASSIFICATION OF NONSIMPLE GRAPH $C^*$-ALGEBRAS


Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
E-mail address: eilers@math.ku.dk

Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA
E-mail address: tomforde@math.uh.edu