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Restriction of Odd Degree Characters of $S_n$

Christine BESSENRODT †, Eugenio GIANNELLI ‡ and Jørn B. OLSSON §

† Institute for Algebra, Number Theory and Discrete Mathematics, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
E-mail: bessen@math.uni-hannover.de

‡ Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WA, United Kingdom
E-mail: eg513@cam.ac.uk

§ Department of Mathematical Sciences, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark
E-mail: olsson@math.ku.dk

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sém. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f_k^n(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f_k^n : \text{Irr}_2(S_n) \longrightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f_k^n$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq n$. Let $v_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: *For which $n$ and $k$ is $f^n_k$ surjective?* The authors showed that $f^n_k$ is surjective whenever $2^n$ is a binary digit of $n$, and they observed that otherwise $f^n_k$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right)$.

- If $k = 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f^n_k$. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2k})$ we define the set

$$\mathcal{E}(\psi, 2^k) = \{ \chi \in \text{Irr}_{2^k}(\mathfrak{S}_n) \mid f^n_k(\chi) = \psi \},$$

and set $e(\psi, 2^k) = |\mathcal{E}(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f^n_k$ are regular on their images. This means that for any $\psi$ in the image of $f^n_k$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f^n_k$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: *when is $f_k f_\ell = f_\ell f_k$?* or more specifically: *when is $f^n_{k-2} f^n_k = f^n_{\ell-2} f^n_\ell$?* In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6, k = 0, \ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

### 2 Notation and background

Let $n$ be a natural number. We let $\text{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of $\mathfrak{S}_n$ and $\mathcal{P}(n)$ the set of partitions of $n$. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(\mathfrak{S}_n)$. We say then that $\lambda$ labels $\chi^\lambda$. We denote by $\text{Irr}_{2^k}(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of odd degree. If $\chi^\lambda \in \text{Irr}_{2^k}(\mathfrak{S}_n)$ we say that $\chi^\lambda$ is an *odd character*, we call $\lambda$ an *odd partition* of $n$ and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let $n, k$ be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f^n_k : \text{Irr}_{2^k}(\mathfrak{S}_n) \to \text{Irr}_{2^k}(\mathfrak{S}_{n-2k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f^n_k(\chi^\lambda) = \chi^\mu \iff \mu \vdash_o n - 2^k$$

can be obtained from $\lambda \vdash_o n$ by removing a $2^k$-hook.

Correspondingly we write (by abuse of notation) $f^n_k(\lambda) = \mu$. In fact when $\lambda$ is odd, there is only one $2^k$-hook of $\lambda$ whose removal leads again to an odd partition; we will refer to such a hook as an *odd hook* of $\lambda$. This combinatorial description of $f^n_k$ will be used throughout this paper, and we will regard $f^n_k$ also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set $e(\mu, 2^k) = e(\chi^\mu, 2^k)$. 


We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then \( Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1}) \) is an \( e \)-tuple of partitions satisfying \( n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i| \). It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let \( H_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( H(Q_e(\lambda)) = \bigcup_{i=0}^{e-1} H(\lambda_i) \).

As explained in [6, Theorem 3.3], there is a bijection between \( H_e(\lambda) \) and \( H(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \).

The \( k \)th row \( Q_2^{(k)}(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2(\lambda_i^{(k)}) \), \( 0 \leq i \leq 2^k - 1 \).

The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda_0^{(0)} \) itself, row 1 contains the partitions \( \lambda_0^{(1)}, \lambda_1^{(1)} \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2(\lambda_i^{(k)}) = (\lambda_{2i}^{(k+1)}, \lambda_{2i+1}^{(k+1)}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2^{(k)}(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_{2^k}(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \). This is a table containing the following \( k+1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda) \), \( j = 0, \ldots, k-1 \), and in addition the row \( Q_2^{(k)}(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k > 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \), because the rows \( C_2^{(l)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2^{(k)}(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \models n - 2^k \) and \( \mu \models n \). The following are equivalent.

\begin{enumerate}
    \item \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.
    \item The \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \), \( \lambda_i^{(k)} \) is obtained from \( \mu_i^{(k)} \) by removing a 1-hook.
\end{enumerate}

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_2^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu_i^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda_i^{(k)} \) being obtained by removing the 1-hook \( H_k \) from \( \mu_i^{(k)} \) (by repeated applications of [6, Theorem 3.3]). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k-1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores.

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let \( c_2^{(k)}(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4** ([5]). Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^{(k)}(\lambda) \leq 1 \) for all \( \leq k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^{(k)}(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q^{(k)}(\lambda) = (\lambda^{(k)}_i)$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.

(ii) The partitions $\lambda^{(k)}_i$, $0 \leq i \leq 2^k - 1$, are all odd.

(iii) The numbers $|\lambda^{(k)}_i|$, $0 \leq i \leq 2^k - 1$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 0} |\lambda^{(k)}_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. 

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = (5, 4, 2^2, 1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the 2-data $D^{(2)}_2(\lambda)$. The 2-core is $C^2_2(\lambda) = (1)$, giving $C^{(0)}_2(\lambda) = (((1)))$. Furthermore, the 2-quotient is $Q_2(\lambda) = (((2^2, 1^2), (1)))$, and computing the 2-cores $C^2_2((2^2, 1^2)) = (0), C^2_2((1)) = (1)$, we obtain the next row: $C^{(1)}_2(\lambda) = ((0), (1))$. The 2-quotients are $Q_2((2^2, 1^2)) = (((1^2), (1))), Q_2((1)) = ((0), (0))$; hence the final row of the 2-data table is obtained as $Q^{(2)}_2(\lambda) = (((2^2), (1), (0), (0))$.

We visualize $D^{(2)}_2(\lambda)$ like this:

\[
\begin{align*}
C^{(0)}_2(\lambda): & \quad (1) \\
C^{(1)}_2(\lambda): & \quad (0) \quad (1) \\
Q^{(2)}_2(\lambda): & \quad (1^2) \quad (1) \quad (0) \quad (0)
\end{align*}
\]

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition $Q_2(\lambda)$ being replaced by $(0)$. Thus, removing the corresponding 4-hook of $\lambda$ we obtain the odd partition $\mu = (3, 2^2, 1^2) \vdash 11$ with the property that $D^{(2)}_2(\lambda)$ and $D^{(2)}_2(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the 2-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Then there are exactly $\prod_{m \leq k} 2^m$ odd partitions $\lambda$ of $n$ with $Q^{(k)}_2(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row $m$ of the $k$-data table to be $(1)$, for each $m < k$ such that $2^m \leq n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f^n_k$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f^n_k$ is surjective) and to show that the maps $f^n_k$ are always regular on their image (see Corollary 3.8).
Definition 3.1. Let $d \geq 0$. We call an odd partition $\lambda$ $d$-good, if 

(i) $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
(ii) $C_{2d}(\lambda)$ is a hook partition.

Let us remark that condition (i) may be reformulated as

(i*) $\nu_2(|\lambda| + 1) = d$.

In particular, if $\lambda$ is $d$-good, then $|\lambda|$ is odd if and only if $d > 0$.

The relevance of $d$-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let $\lambda \vdash_o n$. Let $d = \nu_2(n + 1)$. Then $e(\lambda, 1) \neq 0$ if and only if $\lambda$ is $d$-good. In this case, $e(\lambda, 1) = 1$ if $d = 0$, and $e(\lambda, 1) = 2$ if $d > 0$.

Lemma 3.3. Let $\lambda$ be an odd partition, and let $d \geq 0$. Then the following hold.

(1) For $d \leq 2$, $\lambda$ is $d$-good if and only if $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
(2) If $\lambda$ is $d$-good, then $C_{2d}(\lambda)$ is a partition of $2^d - 1$.

Proof. If the odd partition $\lambda$ is $d$-good, then $|\lambda| = (2^d - 1) + m$ where the binary digits of $m$ are at least $2^{d+1}$. The hooks of $\lambda$ corresponding to the binary digits of $m$ may be decomposed into $2^d$-hooks and thus do not contribute to $C_{2d}(\lambda)$. Thus $|C_{2d}(\lambda)| = 2^d - 1$. This shows (2).

Let $d = 0, 1, 2$ we have $|C_{2d}(\lambda)| = 0, 1$ and 3, respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows.

Definition 3.4. If $2^k \leq n$, we define $d(n, k) = \nu_2\left(\lfloor \frac{n}{2^k} \rfloor \right)$. Thus $d(n, k)$ is the smallest integer $d \geq 0$ satisfying the condition $2^{k+d} \subseteq n$. In particular, $d(n, k) = 0$ if and only if $2^k \subseteq n$. Moreover, we may write $\lfloor \frac{n}{2^k} \rfloor = 2^d(n, k) + m(n, k)$ where $2^{d(n, k)} + 1 \mid m(n, k)$.

As mentioned in the introduction, the results in [3] show that $f_k^n$ is a surjective $(2^k$-to-1$)$-map whenever $2^k \subseteq n$, i.e., $d(n, k) = 0$. In the spirit of [1, Theorem 2], we now give a characterization of the image of the map $f_k^n$ for all $n$, $k$ such that $2^k < n$.

Theorem 3.5. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a $d(n, k)$-good partition in the $k$th row of $Q_2(\lambda)$. In this case, $e(\lambda, 2^k) = 2^k$ if $d(n, k) = 0$, and $e(\lambda, 2^k) = 2$ if $d(n, k) > 0$.

Proof. If $k = 0$ then the statement follows from Lemma 3.2. Hence assume that $k \geq 1$. Let $d = d(n, k)$. By assumption $\lfloor \frac{n}{2^k} \rfloor = 2^d + m$, where the binary digits of $m$ are at least $2^{d+1}$. Thus $\lfloor \frac{n - 2^k}{2^k} \rfloor = (2^d + 1) + m$.

Suppose first that $e(\lambda, 2^k) \neq 0$ and that $\mu \vdash_o n$ satisfies $f_k(\mu) = \lambda$. From Remark 2.1 and Lemma 2.3 we get that there exists an $i \in \{0, 1, \ldots, 2^k - 1\}$ such that $f_0(\mu^{(k)}_i) = \lambda^{(k)}_i$. Since $\mu^{(k)}_i$ and $\lambda^{(k)}_i$ are odd, we get $e(\lambda^{(k)}_i, 1) \neq 0$. We have that $|\lambda^{(k)}_i|$ and $|\mu^{(k)}_i|$ are both 2-disjoint with $m_1 := \sum_{j \neq i} |\lambda^{(k)}_j| = \sum_{j \neq i} |\mu^{(k)}_j| \subseteq 2 \lfloor \frac{n - 2^k}{2^k} \rfloor$, by Theorem 2.5. Since $m_1 \subseteq 2 \lfloor \frac{n - 2^k}{2^k} \rfloor$ and $m_1 \subseteq 2 \lfloor \frac{n}{2^k} \rfloor$, we get $m_1 \subseteq 2 m$. Thus $|\lambda^{(k)}_i| = (2^d + 1) + m_2$ and $|\mu^{(k)}_i| = 2^d + m_2$, where $m_2 = m - m_1 \subseteq 2 m$.

In particular $\nu_2(|\lambda^{(k)}_i| + 1) = \nu_2(|\mu^{(k)}_i|) = d$. Then Lemma 3.2 shows that $\lambda^{(k)}_i$ is $d$-good.

Conversely, if $\lambda^{(k)}_i$ is a $d$-good partition for some $i \in \{0, 1, \ldots, 2^k - 1\}$, then there exists a $\mu^* \vdash_o |\lambda^{(k)}_i| + 1$ such that $f_0(\mu^*) = \lambda^{(k)}_i$, by Lemma 3.2. We let $\mu$ be the partition where the $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that $\mu^{(k)}_i = \mu^*$. Since $\lambda$ is odd and $\lambda^{(k)}_i$ is $d$-good,
we know that \( |\lambda_i^{(k)}| = (2^d - 1) + m' \) where \( m' \subseteq m \), and \( |\lambda_j^{(k)}| \subseteq 2m - m' \) for all \( j \neq i \). Hence \( |\mu^*| = |\lambda_i^{(k)}| + 1 = 2^d + m' \) is 2-disjoint from all \( |\lambda_j^{(k)}| \), \( j \neq i \). Thus \( \mu \) is an odd partition of \( n \) by Theorem 2.5, and \( f_k(\mu) = \lambda \) by Lemma 2.3 and Remark 2.1.

We conclude that \( e(\lambda, 2^k) = \sum_{\lambda_i^{(k)} \text{d-good}} e(\lambda_i^{(k)}, 1) \). If \( d = 0 \) then \( \left\lceil \frac{n-2^k}{2^d} \right\rceil \) is even. This implies that all \( \lambda_i^{(k)} \) are of even cardinality and thus d-good. Thus \( e(\lambda_i^{(k)}, 1) = 1 \) for all \( i \), and we get \( e(\lambda, 2^k) = 2^k \). If \( d > 0 \) there is exactly one \( \lambda_i^{(k)} \) in \( Q_2^{(k)}(\lambda) \) of odd cardinality. Only this \( \lambda_i^{(k)} \) may be d-good and then \( e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2 \). Otherwise \( e(\lambda, 2^k) = 0 \). \( \blacksquare \)

**Corollary 3.6.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \), and let \( d = \nu_2(\left\lceil \frac{n}{2^k} \right\rceil) \). Let \( \lambda \vdash_o n - 2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a partition \( \lambda_i^{(k)} \) in the \( k \)th row of \( Q_2(\lambda) \) such that \( |\lambda_i^{(k)}| = 2^d - 1 \mod 2^d + 1 \), and \( C_{2d}(\lambda_i^{(k)}) \) is a hook partition. In this case, \( e(\lambda, 2^k) = 2^k \) if \( d = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d > 0 \).

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \).

- If \( k = 0 \) then \( f_k^n \) is surjective if and only if \( d(n, k) \leq 2 \).
- If \( k > 0 \) then \( f_k^n \) is surjective if and only if \( d(n, k) \leq 1 \).

**Proof.** By Theorem 3.5, \( f_k^n \) is surjective if and only if for all \( \lambda \vdash_o n - 2^k \) we have that the \( k \)th row of \( Q_2(\lambda) \) contains a \( d(n, k) \)-good partition \( \lambda_i^{(k)} \). By Theorem 2.5 and Definition 3.4, for any \( \lambda \vdash_o n - 2^k \) we have \( \sum_{j \geq 0} |\lambda_j^{(k)}| = \left\lceil \frac{n-2^k}{2^d} \right\rceil = (2^{d(n,k)} - 1) + m(n, k) \).

If \( k = 0 \) then \( Q_2^{(0)}(\lambda) \) contains only \( \lambda = \lambda_0^{(0)} \). Hence \( f_k^n \) is surjective if and only all odd partitions of \( n - 1 \) are \( d(n, 0) \)-good. By Lemma 3.5(1), the latter condition holds when \( d = d(n, 0) \leq 2 \). On the other hand, if \( d = \nu_2(n) > 2 \), then \( \lambda = (n - 5, 2, 2) \) is an odd partition of \( n - 1 \) by Theorem 2.5, but \( C_8(\lambda) = (3, 2, 2) \) is not a hook, and hence \( C_{2d}(\lambda) \) is not a hook. So \( \lambda \) is not \( d \)-good, and thus \( f_k^n \) is not surjective.

Now assume \( k \geq 1 \). Then \( Q_2^{(k)}(\lambda) \) contains at least two odd partitions. If \( d(n, k) \geq 2 \) then any \( d(n, k) \)-good partition \( \mu \) satisfies \( 3 \subseteq 2^{d(n,k)} - 1 \subseteq |\mu| \). Write \( \left\lceil \frac{n-2^k}{2^d} \right\rceil = 1 + m_1 \) where \( m_1 \) is even. Applying Remark 2.7, take any \( \lambda \vdash_o n - 2^k \) such that \( |\lambda_0^{(k)}| = 1 \) and \( \lambda_i^{(k)} \) is an odd partition with \( |\lambda_1^{(k)}| = m_1 \). Then no partition in \( Q_2^{(k)}(\lambda) \) is \( d(n,k) \)-good. Thus \( f_k^n \) is not surjective. On the other hand, if \( d(n, k) = 0 \) then \( 2^k \subseteq n \) and \( f_k^n \) is surjective [3, Proposition 4.5]. If \( d(n, k) = 1 \) then \( \left\lceil \frac{n-2^k}{2^d} \right\rceil = 1 + m(n, k) \), where \( 4 | m(n, k) \). Thus any \( Q_2^{(k)}(\lambda) \) contains a partition with odd cardinality; this partition is \( 1 \)-good, by Lemma 3.3. Again \( f_k^n \) is surjective. \( \blacksquare \)

It is an immediate consequence of Theorem 3.5 that \( f_k^n \) is regular on its image for all relevant choices of \( n, k \) such that \( 2^k < n \). We have:

**Corollary 3.8.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \); set \( d = \nu_2(\left\lceil \frac{n}{2^k} \right\rceil) \). Let \( \lambda \vdash_o n - 2^k \). Then

\[
e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k \text{th row of } Q_2(\lambda) \text{ contains a } d \text{-good partition}; \\ 0 & \text{otherwise.} \end{cases}
\]
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n, k) = \nu_2 \left( \frac{n}{2^k} \right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6 \), \( d(6, 2) = 0 \). Thus \( e(2, 4) = 4 \). The odd 4-extensions of \( (2) \) are \((6), (3^2), (2^2, 1^2), (2, 1^4)\). For \( n = 10 \), \( d(10, 2) = 1 \). In this case, \( e(\lambda, 4) = 2 \) for all odd partitions \( \lambda \) of 6. For instance, the odd 4-extensions of \((6)\) are \((10)\) and \((6, 3, 1)\). For \( n = 19 \), \( d(19, 2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \), \( 15 \) there is no 2-good partition in \( Q_2^{(2)}(\lambda) \), hence \( e(\lambda, 4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \((n; k, \ell) \in \mathcal{T}\) if for all \( \lambda \vdash_o \) \( n \) we have \( f_k f_\ell(\lambda) = f_\ell f_k(\lambda) \). Otherwise we write \((n; k, \ell) \in \mathcal{F}\).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0, 1)\)

\[(n; k, \ell) \in \mathcal{F} \text{ if and only if } \ell < t \text{ and } 2^k \leq m.\]

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \((n; k, t) \in \mathcal{T}\).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \((n; k, \ell) \in \mathcal{T} \) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq_2 a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \((2^t - b, 1^b)\), \( 0 \leq b \leq 2^t - 1 \), of degree \((2^t-1)\). Hence for \( k \in \{0, 1, \ldots, t-1\} \) we have

\[ f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \nsubseteq_2 b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq_2 b. \end{cases} \]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k(\lambda) = f_k f_\ell(\lambda) \).

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \((n; k, \ell) \in \mathcal{T}\).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda \vdash_o n \). Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{\ell-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \( |\lambda_0| \) and \( |\lambda_1| \) are 2-disjoint binary subsums of \( \lfloor \frac{n}{2^t} \rfloor \), so one of them contains \( 2^{t-1} \), say \( |\lambda_0| \); then \( |\lambda_1| \leq \lfloor \frac{m}{2^{t-1}} \rfloor < 2^{k-1} < 2^{\ell-1} \). Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[ Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1). \]
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence
\[ Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1} f_{k-1}(\lambda_0), \lambda_1)). \]
In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^k - 1$. Also $|\lambda_0| + |\lambda_1| = \left\lfloor \frac{m}{2} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{m}{2} \right\rfloor$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$, furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lfloor \frac{m}{2} \right\rfloor < 2^k - 1$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. \hfill

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^\ell + m$ with $0 < m < 2^\ell$. If $0 < \ell < t$ then $(n; 0, \ell) \in F$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

**Case 1:** $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^\ell - m$. The $(1,1)$-hook length of $\lambda$ is $2^{\ell+1}$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that
\[ f_0(\lambda) = (m, m, 1^{a-1}). \]
Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now
\[ f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^\ell, 1^{a-1}) \]
since $(m, m, 1^{a-1-2^\ell})$ and $(m - 1, m - 2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,
\[ f_\ell(\lambda) = (m - 1, m - 2^{\ell-1}, 1^a). \]
Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then
\[ f_0(f_\ell(\lambda)) = f_0(m - 1, m - 2^{\ell-1}, 1^a) = (m - 1, m - 2^\ell, 1^a). \]
This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $(m - 1, m - 2^{\ell-1}, 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 2:** $m < 2^\ell$. Consider the partition $\lambda = (n - 2^\ell, m + 1, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now
\[ f_0(\lambda) = (n - 2^\ell, m, 1^a) \]
since the other candidates do not have hooks of length $2^\ell$. Then
\[ f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu, \]
where \( \mu \) is obtained from \( f_0(\lambda) \) by removing a \( 2^\ell \)-hook in the first row. (There are only hooks of length \( < 2^\ell \) in the other rows.) In fact, \( \mu = (n - 2^{\ell+1}, m, 1^a) \) since \( n - 2^{\ell+1} \geq n - 2^\ell = m \). Thus \( f_\ell(f_0(\lambda)) \) has at least 2 parts. On the other hand
\[
f_\ell(\lambda) = (n - 2^\ell)
\]

since this odd partition is obtained from the odd partition \( \lambda \) by removing a \( 2^\ell \)-hook (the one in (2,1)). It follows that
\[
f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)
\]

and again \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

**Case 3:** \( m = 2^\ell \). Then \( n = 2^\ell + 2^\ell \). If \( \ell \geq 2 \) then choose \( \lambda = (2^\ell, 2^\ell - 1, 1) \). The \( (1,2) \)-hook length of \( \lambda \) is \( 2^\ell \); thus \( \lambda \) is an odd partition since removing this \( 2^\ell \)-hook gives an odd partition \( (2^\ell - 2, 1, 1) \) of \( 2^\ell \). We have \( f_0(\lambda) = (2^\ell, 2^\ell - 2, 1) \) since the other candidates are not odd. Then
\[
f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).
\]
The \( (2,1) \)-hook length of \( \lambda \) is \( 2^\ell \), so \( f_\ell(\lambda) = (2^\ell) \) and
\[
f_0(f_\ell(\lambda)) = (2^\ell - 1),
\]
showing \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

On the other hand, if \( \ell = 1 \) then choose \( \lambda = (2^\ell - 2, 2, 2) \) which is both in \( (6; 0, 1) \). Since \( t \geq 3 \), it is now easy to show that \( f_1(f_0(\lambda)) = (2^t - 4, 2, 1) \). On the other hand we see that \( f_0(f_1(\lambda)) \) is a hook partition of \( 2^t - 1 = n - 3 \) and therefore is not equal to \( f_1(f_0(\lambda)) \).

**Lemma 4.6.** If \( (n; k, \ell) \in \mathcal{F} \) then also \( (2n; k + 1, \ell + 1) \in \mathcal{F} \) and \( (2n + 1; k + 1, \ell + 1) \in \mathcal{F} \).

**Proof.** Let the odd partition \( \mu \) of \( n \) satisfy \( f_k f_\ell(\mu) \neq f_\ell f_k(\mu) \). Let \( \lambda \) be a partition of \( 2n \) or \( 2n + 1 \) having 2-quotient \( Q_2(\lambda) = (\mu, (0)) \). Then \( \lambda \) is odd, by Theorem 2.5. We have
\[
Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)),
\]
so that \( f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda) \).

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on \( k \geq 0 \). If \( k = 0 \), then the statement follows from Lemma 4.5. Let \( k > 1 \) and suppose that the assertion is true up to and including \( k - 1 \). To show that \( (n; k, \ell) \in \mathcal{F} \) it suffices to prove \( (\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) \in \mathcal{F} \), by Lemma 4.6. We are assuming \( n = 2^\ell + m, 0 \leq m < 2^\ell, 0 \leq k < \ell \leq t \) and \( 2^k + 2^\ell \leq n \). This implies \( \lfloor \frac{n}{2} \rfloor = 2^{\ell-1} + \left\lfloor \frac{m}{2} \right\rfloor, 0 \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{\ell-1} \) and \( 2^{k-1} + 2^{\ell-1} \leq \left\lfloor \frac{n}{2} \right\rfloor \). We may apply the inductive hypothesis to get \( (\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) \in \mathcal{F} \), and then \( (n; k, \ell) \in \mathcal{F} \) except when \( (\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) = (6; 0, 1) \). In that case we are considering \( (12; 1, 2) \) or \( (13; 1, 2) \) which are both in \( \mathcal{F} \), by direct computation (consider for example \( (6, 4, 2) \in \mathcal{F} \)).

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