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Visualizing and representing the evolution of topological features

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Abstract

Simplicial complexes are discrete representations of topological spaces that are practical for computational studies. The first three Betti-numbers (indicating the number of components, tunnels and voids), as well as the topological persistence of each such feature, is well-defined and can be efficiently computed for simplicial complexes embedded in 2D and 3D [1, 2].

We introduce a novel representation of the evolution of topological features in simplicial complexes using so-called tunnel-trees in 2D and void-trees in 3D. This new representation makes it possible to analyze topological evolution by applying tools for analysis of binary trees. Furthermore it supplies a new method for visualizing topological evolution.

Introduction

A simplicial complex, \( K \), is a set of simplices where any face of a simplex in \( K \) is also in \( K \) and the intersection of two simplices in \( K \) is either empty or a face of both simplices. Delfinado and Edelsbrunner [1] define a filter to be a sequence of simplices, \( \sigma^1, \sigma^2, \ldots, \sigma^n \), where \( K_i = \{ \sigma^1, \sigma^2, \ldots, \sigma^i \} \) is a simplicial complex for any choice of \( i \) (see left part of Figure 1). The filter represents the evolution of a simplicial complex and will be the focus of the methods described here. The topological features of a complex can be described using the Betti-numbers, \( \beta_d \), which indicate the rank of the \( d \)th homology group. The first three Betti-numbers \( (\beta_0, \beta_1, \beta_2) \) can be interpreted more intuitively as the number of components, holes, and voids respectively.

A \( O(na(n)) \)-time algorithm exists to calculate the evolution of \( \beta_d \) as a simplicial complex is grown using a filter [1]. This method identifies each \( k \)-simplex, \( \sigma^i \), as either positive if it creates a new \( k \)-cycle and thereby increases \( \beta_k \), or negative if it changes a \( k \)-cycle into a \( k \)-boundary and thereby decreases \( \beta_k - 1 \). For each positive \( k \)-simplex, \( \sigma^i \), the negative \((k + 1)\)-simplex, \( \sigma^{i+1} \), is responsible for turning the \( k \)-cycle, created by \( \sigma^i \), into a \( k \)-boundary which can be efficiently identified [2]. The difference between the indices of such two simplices is defined to be the persistence of the \( k \)-cycle represented by \( \sigma^i \).

Tunnel- and void-trees

One interesting observation about tunnels in simplicial complexes embedded in 2D is that, often, when a positive 1-simplex (edge) is added to the complex, it splits one tunnel in two. If the empty space around the complex is considered a bounding tunnel, then every positive edge will split an existing tunnel in two. Similarly, if the entire space around a simplicial complex embedded in 3D is considered a bounding complex, then a positive 2-simplex (triangle) always splits an existing void in two.

Based on this observation we define a tunnel-tree (or \( \beta_1 \)-tree) of a 2D filter to be a binary tree where each node represents a distinct tunnel (see right part of Figure 1). The root is the bounding tunnel, and the leaves are triangular tunnels that will not be split further. With each node we associate the positive edge that represents the tunnel, \( \epsilon(n) \), and with each leaf, we associate the negative triangle that fills this tunnel, \( \tau(n) \). The tunnel-tree is ordered such that for any node \( n \), the triangle of the rightmost leaf, \( \tau(\text{Tree-Max}(n)) \), is the triangle that ‘destroys’ \( \epsilon(n) \) and hence determines its persistence. A void-tree (or \( \beta_2 \)-tree) of a 3D filter is defined in a similar fashion, only with positive triangles as nodes and negative tetrahedra as leaves.

A \( \beta_k \)-tree is constructed by running through the filter backwards as shown in Algorithm 1. Leaves are created when a negative \((k + 1)\)-simplex is encountered and the roots of leaves are connected when positive \( k \)-simplices are encountered.

Algorithm 1 Build a \( \beta_k \)-tree given a filter

1: Create a 'bounding node', \( \sigma^0_b \)
2: for \( i = n \) to 1 do
3: if \( \sigma^i \) is a negative \((k + 1)\)-simplex then
4: Create a new node, \( n \), and set \( \tau(n) \leftarrow \sigma^i \)
5: else if \( \sigma^i \) is a positive \( k \)-simplex then
6: \((n_0, n_1) \leftarrow \text{Nodes of the two } (k + 1)\text{-simplices adjacent to } \sigma^i \)
7: \((n_0, n_1) \leftarrow (\text{Root}(n_0), \text{Root}(n_1)) \)
8: Swap \( n_0 \) and \( n_1 \) if \( \tau(\text{Tree-Max}(n_0)) \) is younger than \( \tau(\text{Tree-Max}(n_1)) \)
9: Create a new node \( n \) with \( n.left \leftarrow n_0, n.right \leftarrow n_1 \), and \( \epsilon(n.left) \leftarrow \sigma^i \)
10: end if
11: end for
12: return Root\( (n_b) \)

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In line 4, the \((k + 1)\)-simplex can be associated with its node using a hash-map. This ensures that locating the nodes of adjacent \((k + 1)\)-simplices in line 6 can be performed in constant time. In line 6, if one of the \((k + 1)\)-simplices adjacent to \(\sigma^i\) is not defined then the bounding node \(n_b\) is used instead. If \(\sigma^i\) has no adjacent \((k + 1)\)-simplices then a new node is created for \(n_0\), and \(n_1\) is set to \(n_b\). Line 8 guarantees that the youngest simplex in a subtree can always be found by going to the far right in the tree using TREE-MAX.

A \(\beta_k\)-tree may be arbitrarily unbalanced, so a straightforward implementation will run in \(O(n^2)\) time worst case. The TREE-MAX-method can be improved to \(O(1)\) time by maintaining the maximum of each sub-tree as they are constructed. A data structure similar to disjoint-sets can be used to make the ROOT method run in \(O(\alpha(n))\)-time, so the entire method runs in \(O(n\alpha(n))\) worst case time.

**Applications**

One attractive property of \(\beta_k\)-trees is that they give an alternative representation of the topological evolution of a filter. This can be used in several ways.

First, the fact that simplices in the subtree of a particular node will tend to be spatially close to each other gives rise to a new definition of local persistence. A particular edge, representing a tunnel, might be deemed particularly persistent if its subtree contains more than a certain number of nodes. Such a definition of persistence will not be affected by the addition of simplices outside the tunnel.

Using a Delaunay complex and the radius of the smallest empty circumcircle to generate an \(\alpha\)-filter [3], the arrangement of a particular sub-tree also gives an indication of the shape of the corresponding feature. For instance, a node with an unbalanced sub-tree indicates a tunnel that is narrowing, whereas a balanced node indicates a constant width.

For some applications, a tree might be a better visualization of the topological evolution than e.g. \(k\)-triangles [2]. The above mentioned properties of locality can be computationally analyzed, but they can also be derived simply by inspecting \(\beta_k\)-trees. The length of edges in the tree can furthermore be scaled to reflect the difference in birth time of the \(\epsilon(n)\) simplices.

Another interesting property of \(\beta_k\)-trees is that all \((k+1)\)-simplices within a particular tunnel/void are easily identified by locating the node in the tree with the desired \(\epsilon(n)\) and then collecting all leaves in the subtree using any tree-traversal method. In this manner the area of tunnels/volume of voids, for instance, is easily calculated.

Finally, any analysis method that works on trees is now applicable to topological evolutions. For instance the topology of two point-sets can be compared by finding the tree-edit-distance between the tunnel-trees (or void-trees) of their respective \(\alpha\)-filters.

**References**

