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Large excursions and conditioned laws for recursive sequences generated by random matrices

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Abstract

We determine the large exceedance probabilities and large exceedance paths for the matrix recursive sequence

\[ V_n = M_n V_{n-1} + Q_n, \quad n = 1, 2, \ldots, \]

where \( \{M_n\} \) is an i.i.d. sequence of \( d \times d \) random matrices and \( \{Q_n\} \) is an i.i.d. sequence of random vectors, both with nonnegative entries. Early work on this problem dates to Kesten’s (1973) seminal paper, motivated by an application to multi-type branching processes. Other applications arise in financial time series modeling (connected to the study of the GARCH(\( p, q \)) processes) and in physics, and this recursive sequence has also been the focus of extensive work in the recent probability literature. In this work, we characterize the distribution of the first passage time \( T_A := \inf \{ n : V_n \in uA \} \), where \( A \) is a subset of the nonnegative quadrant in \( \mathbb{R}^d \), showing that \( T_A/ u^\alpha \) converges to an exponential law. In the process, we also revisit and refine Kesten’s classical estimate, showing that if \( V \) has the stationary distribution of \( \{V_n\} \), then \( P(V \in uA) \sim C_A u^{-\alpha} \) as \( u \to \infty \), providing, most importantly, a new characterization of the constant \( C_A \). Finally, we describe the large exceedance paths via two conditioned limit laws. In the first, we show that conditioned on a large exceedance, the process \( \{V_n\} \) follows an exponentially-shifted Markov random walk, which we identify, thereby generalizing results for classical random walk to matrix recursive sequences. In the second, we characterize the empirical distribution of \( \{\log |V_n| - \log |V_{n-1}|\} \) prior to a large exceedance, showing that this distribution converges to the stationary law of the exponentially-shifted Markov random walk.

1 Introduction

The goal of this paper is to describe the extremal behavior, tail asymptotics, and conditioned path properties of the matrix recursive sequence

\[ V_n = M_n V_{n-1} + Q_n, \quad n = 1, 2, \ldots, \quad V_0 = v \in \mathbb{R}_d^+ \tag{1.1} \]

where \( \{M_n\} \) is an i.i.d. sequence of \( d \times d \) random matrices with nonnegative entries, \( \{Q_n\} \) is an i.i.d. sequence of nonnegative random vectors, and \( \mathbb{R}_d^+ := [0, \infty)^d \) denotes the nonnegative quadrant in \( d \)-dimensional Euclidean space.

Motivated by branching processes in random environments with immigration, as considered in Solomon (1972, 1975), the matrix recursive sequence (1.1) was originally studied in the fundamental paper of Kesten (1973). If \( E[\log \|M_1\| + \log |Q_1|] < \infty \) and the upper Lyapunov exponent is negative, i.e.

\[ \lim_{n \to \infty} \frac{1}{n} \log \|M_1 \cdots M_n\| < 0, \]

then it is readily verified that the law of

\[ V := \sum_{k=1}^\infty M_1 \cdots M_{k-1} Q_k \]

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is the unique stationary distribution for the Markov chain \( \{V_n\} \) (see e.g. Bougerol and Picard (1992)). Then under a Cramér-type condition stating that \( \lambda(\alpha) = 1 \) for some \( \alpha > 0 \), where

\[
\lambda(\theta) := \lim_{n \to \infty} \left( \mathbb{E} \left[ \|M_n \cdots M_1\|^\theta \right] \right)^{1/n}
\]

and \( \| \cdot \| \) denotes operator norm, Kesten studied \( \mathbb{P}(\langle w, V \rangle > u) \) as \( u \to \infty \) for \( w \in \mathbb{R}^d_+ \). It is shown in Kesten (1973) that under appropriate moment and irreducibility conditions,

\[
\mathbb{P}(\langle w, V \rangle > u) \approx \mathcal{C} u^{-\alpha} \quad \text{as} \quad u \to \infty
\]  

(1.2)

for a certain constant \( \mathcal{C} \). Historically, this estimate resolved a conjecture by Spitzer, verifying that \( V \) lies in the domain of attraction of a stable law.

Recently, there has been a renewed interest in Kesten’s estimate. The asymptotics in (1.2) have been shown to characterize the stationary tail decay in the GARCH\((p, q)\) financial time series models or, similarly, the ARMA\((p, q)\) processes with random coefficients; cf. de Haan et al. (1989), Mikosch (2003). The process (1.1) is also relevant for the study of random walk in random environment (cf., e.g., Kesten et al. (1975), Wang (2013)), and in a variety of other problems related to branching processes and Mandelbrot cascades; cf. Guivarc’h (1990), Liu (2000), Buraczewski, Damek, and Mikosch (2016) and references therein. Furthermore, in recent years, the scope of Kesten’s method has broadened to include more general fixed point equations in \( \mathbb{R} \): namely equations of the form

\[
V = F(V),
\]

(1.3)

where \( F: \mathbb{R} \to \mathbb{R} \) is a random function independent of \( V \), and \( F(v) \approx Mv \) for large \( v \), where \( M \) is a random variable in \( \mathbb{R} \); cf. Goldie (1991), Mirek (2011), Collamore and Vidyashankar (2013a,b), Alsmeyer (2016). [Here, \( \approx \) denotes equality in distribution.] Moreover, generalizations to Markov-dependent recursive sequences (satisfying different assumptions from the processes we consider here) have been obtained by Roitershtein (2007) and Collamore (2009).

It is natural to ask whether this theory may be extended to reveal more refined path properties of the process \( \{V_n\} \). In particular, the behavior of \( \{V_n\} \) over large excursions may essentially be inferred from that of the Markov random walk \( \{(X_n, S_n) : n = 0, 1, \ldots\} \), where

\[
X_n = \frac{M_n \cdots M_1 v}{|M_n \cdots M_1 v|}, \quad S_n = \log |M_n \cdots M_1 v|,
\]

(1.4)

and \( | \cdot | \) denotes a norm in \( \mathbb{R}^d \) (thus, \( \{X_n\} \) describes the directional component of the matrix product \( M_n \cdots M_1 v \), and \( \{S_n\} \) describes its radial growth). Note that \( e^{S_n} X_n \) corresponds with \( V_n \) when \( Q = 0 \). While the rough equivalence between \( \{V_n\} \) and \( \{e^{S_n} X_n\} \) has been utilized by numerous authors, including Kesten (1973), the correspondence between these processes has typically only been employed to obtain estimates such as (1.2), and not to characterize more detailed path properties. In contrast, our approach will be to quantify this discrepancy using Markov nonlinear renewal theory, as developed by Melfi (1992, 1994), yielding—after accounting for the small-time behavior—that the process \( \{V_n\} \) is closely approximated by \( \{e^{S_n} X_n\} \) in a manner which we characterize mathematically. Consequently, it is natural to expect that, over a large excursion, the random walk structure inherent in \( \{(X_n, S_n)\} \) may be exploited to yield deeper characteristics of the process \( \{V_n\} \) which mimic known properties of Markov random walk. Following this approach, we shall reexamine Kesten’s estimate, then extend the approach to obtain related asymptotic results relevant in extreme value theory, and, ultimately, derive path estimates conditioned on a large excursion, showing quantitatively that the path of \( \{V_n\} \) to a large exceedance roughly follows that of \( \{e^{S_n} X_n\} \) in an \( \alpha \)-shifted measure—also known as the exponentially tilted measure or Esscher transform—generalized to the setting of Markov random walk.

We start by revisiting (1.2), establishing that, for an arbitrary set \( A \subset \mathbb{R}^d_+ \) satisfying certain regularity constraints,

\[
\mathbb{P}(V \in uA) \sim \frac{C}{X'(\alpha)} \mathcal{L}_\alpha(A) u^{-\alpha} \quad \text{as} \quad u \to \infty.
\]

(1.5)
for a universal constant $C$ and a measure $\Sigma_n$. In particular, the constant $C$ is now explicitly identified as the $\alpha$th moment of a certain power series derived from $\{(M_n, Q_n)\}$ and the time-reversed products of $\{M_n\}$; see (2.14) and (2.15) below. The formula we obtain can be viewed as a multidimensional extension of the main result in Collamore and Vidyashankar (2013b). (For a related one-dimensional estimate, see also Enriquez et al. (2009).) Roughly speaking, $C$ is obtained by studying the ratio $|V_n|/[M_n \cdots M_1 v]$ along a large excursion, and intuitively, the constant arises by comparing the growth of $|V_n|$ to the product $|M_n \cdots M_1 v|$ along paths where both of these processes diverge; see the discussion in Remark 2.6 below.

Moreover, from (1.5) we immediately conclude that $V$ is multivariate regularly varying, as could only be deduced from (1.2), based on the current literature, with the help of the Cramér-Wold device; see Basrak et al. (2002) and Boman and Lindskog (2009). We emphasize that this additional step is not needed in our approach.

Following a similar approach, we then examine the extremal behavior of $\{V_n\}$. Specifically, letting $A$ be a subset of the nonnegative quadrant and setting $T^A_u = \inf \{n : V_n \in uA\}$, we study the growth rate of $T^A_u$ as $u \to \infty$. We show that

$$\lim_{u \to \infty} \mathbb{P}\left( \frac{T^A_u}{u^\alpha} \leq z \mid V_0 = v \right) = e^{-K\alpha z}, \quad z \geq 0,$$

(1.6)

where $\alpha$ is given as in (1.2) and $K_\alpha$ is a constant which we also characterize, relating this constant explicitly to the prefactor appearing in the asymptotic expression, as $u \to \infty$, for the hitting probability of the set $uA$ of $\{e^{S_u} X_n\}$ and to the constant $C$. As a special case, setting $A = \{x : |x| > 1\}$, we can then conclude that $\{\{V_n\}\}$ belongs to the maximum domain of attraction of the Fréchet distribution. However, it should be emphasized that (1.6) is actually a stronger result, yielding the directional dependence of $\{V_n\}$ and suggesting a natural extension of classical extreme value theory to this multidimensional setting. Note that in our estimate, $T^A_u < \infty$ a.s.; this contrasts from the asymptotics one obtains for perpetuity sequences (i.e., the backward sequences corresponding to (1.1) in $\mathbb{R}$). In that setting, one obtains asymptotics which partly mimic those of random walk; cf. Buraczewski, Collamore, Damek, and Zienkiewicz (2016), which may be compared with random walk estimates such as Lalley (1984) or Collamore (1998). In contrast, (1.6) is qualitatively similar to reflected random walk, and (1.6) can be viewed as an extension, to our setting, of a classical result due to Iglehart (1972). (For maximal segmental sums of random walks, closely related estimates have also been provided by Dembo and Karlin (1991a, b), Karlin and Dembo (1992), and Dembo et al. (1994).) Our result also sharpens earlier work, largely restricted to one-dimensional recursions, due to de Haan et al. (1989), Perfekt (1994, 1997), and Buraczewski, Damek, and Mikosch (2016); cf. Remark 2.10 below.

The key to establishing (1.5) and (1.6) is a proposition, where we study the behavior of $\{V_n\}$ over cycles emanating from, and then returning to, a given set $\mathbb{D} \subset \mathbb{R}^d$. Drawing an analogy with reflected random walk, these returns to $\mathbb{D}$ play the role of Iglehart’s (1972) returns of a reflected random walk to the origin. Letting $\tau$ denote the first return time to $\mathbb{D}$, then for any suitable function $g$ and any $m \in \{1, 2, \ldots\}$, we consider in Proposition 4.1 the limit behavior, as $u \to \infty$, of

$$u^{\alpha} \mathbb{E}\left[ g\left( \frac{V_{T^A_u}}{u}, \ldots, \frac{V_{T^A_u + m}}{u} \right) \mathbf{1}_{\{T^A_u < \tau\}} \mid V_0 = v \right].$$

If $g = 1$, then this represents the rescaled probability that $\{V_n\}$ enters the set $uA$ before returning to $\mathbb{D}$. Moreover, for general $g$, we show that the post-$T^A_u$-process behaves as $\{e^{S_u} X_n\}$, but starting with the stationary overjump distribution. This idea is then extended in the final section of the article to include the path behavior prior to time $T^A_u$ more explicitly, drawing a close analogy to the behavior of the process $\{e^{S_u} X_n\}$ in the $\alpha$-shifted measure.

The motivation of our concluding results is to establish an extension of a well-known estimate for random walk; namely, that a negative-drift random walk satisfying a Cramér-type condition and conditioned to stay positive behaves as its associate (i.e., the random walk in the $\alpha$-shifted measure); cf. Feller (1971), Section XII.6.(d); Bertoin and Doney (1994). Similarly, a negative-drift random walk conditioned to achieve a high barrier at level $u$ will also converge to its associate as the level $u \to \infty$;
We show that for any suitable continuous function $g$, cf. Asmussen (1982). Thus, it is natural to expect that, as $u \to \infty$, the process $\{e^{S_n}X_n\}$ in the $\alpha$-shifted measure, and in Section 2.4, we make this idea precise. As a special case, we then consider the empirical law of $\{\log |V_n| - \log |V_{n-1}|\}$ conditioned on $\{T_u^A < \tau\}$. We show that for any suitable continuous function $g$,

$$
\mathbb{E} \left[ \frac{1}{T_u^A} \sum_{k=1}^{T_u^A} g \left( \log \left( \frac{|V_k|}{|V_{k-1}|} \right) \right) \mid V_0 = v, T_u^A < \tau \right] \to 0 \tag{1.7}
$$

as $u \to \infty$, where $\mathbb{E}^\alpha[\cdot]$ denotes expectation, under stationarity, in the $\alpha$-shifted measure, and $S_1$ is given as in (1.4). Thus, the empirical law of $\{\log |V_n| - \log |V_{n-1}|\}$ converges weakly in $\mathbb{P} (\cdot \mid T_u^A < \tau)$-probability to the distribution, under stationarity, of $S_1$.

We emphasize that we shall develop our limit theorems without the assumption that the Markov chain $\{V_n\}$ is Harris recurrent, and thus—while we shall occasionally draw upon the theory of Harris recurrent chains—our approach will differ markedly from the more classical approach outlined, for example, in Ney and Nummelin (1987). Indeed, the assumption of Harris recurrence is rather unnatural in our setting. Instead, we circumvent this requirement by introducing a smoothing technique, where the sequence $\{Q_{kn}\}$ is “smoothed” for some $k \in \{1, 2, \ldots\}$, thereby ensuring that the resulting process is Harris recurrent, yet the effect of this smoothing is negligible in an asymptotic limit. As this technique could be adapted to other recursive sequences satisfying a stochastic fixed point equation of the form (1.3)—where the assumption of Harris recurrence is also restrictive—this method could potentially be of some general interest. Indeed, rather than assuming Harris recurrence, we shall rely throughout the article on the recently-developed theory of Guivarc’h and Le Page (2016), which exploits spectral gap properties on special function spaces for matrix products under weak regularity conditions. While the theory of Guivarc’h and Le Page is developed for invertible matrices, a formulation for matrices with nonnegative entries, as we consider here, is given in Buraczewski et al. (2014). We now turn to a precise statement of our main results.

## 2 Statement of results

### 2.1 Preliminaries: notation, assumptions, and background

Let $d \geq 1$ and let $M$ be a $d \times d$ random matrix whose entries are nonnegative a.s., and let $Q$ be a random vector in $\mathbb{R}^d$ with nonnegative entries a.s. Let $\mu$ denote the probability law of $(M, Q)$, and let $\mu_M, \mu_Q$ denote the marginal laws of $M$ and $Q$, respectively. Now let $\{(M_n, Q_n) : n \in \mathbb{N}_+\}$ be a sequence of i.i.d. copies of $(M, Q)$ where, here and in the following, $\mathbb{N}_+ := \{1, 2, \ldots\}$ denotes the positive integers. We assume throughout the article that $\{(M_i, Q_i) : i = 1, \ldots, n\}$ are adapted to a given filtration $\{\mathcal{F}_i : i = 1, 2, \ldots\}$.

Then the aim of this paper is to study the extremal properties of the stochastic recursive sequence defined by

$$
V_n = M_n V_{n-1} + Q_n, \quad n = 1, 2, \ldots, \quad V_0 \sim \nu, \tag{2.1}
$$

for a given initial distribution $\nu$, where, unless specifically noted, $\nu$ is concentrated at a deterministic point $v \in \mathbb{R}^d$.

Next we introduce some additional notation, as follows. Let $\mathbb{R}^d$ be endowed with the scalar product $\langle \cdot, \cdot \rangle$ and canonical orthonormal basis $\{e_i : i = 1, \ldots, d\}$. The nonnegative cone in $\mathbb{R}^d$ is defined by

$$
\mathbb{R}^d_+ = \left\{ x \in \mathbb{R}^d : \langle x, e_i \rangle \geq 0 \right\}.
$$

Let $|\cdot|$ denote a norm in $\mathbb{R}^d$, and assume throughout the article that $|\cdot|$ is monotone, i.e., if $x, y \in \mathbb{R}^d_+$ satisfy $y - x \in \mathbb{R}^d_+$, then $|x| \leq |y|$. Let $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ denote the unit sphere and $S^{d-1}_+ := \mathbb{R}^d_+ \cap S^{d-1}$. For any $x \in \mathbb{R}^d$, let $\bar{x}$ denote its projection onto the unit sphere, that is,

$$
\bar{x} \equiv (x) := |x|^{-1}x, \quad x \in \mathbb{R}^d.
$$
For any subspace of $\mathbb{R}^d$, let $\mathcal{B}(\mathcal{J})$ denote the collection of Borel sets on $\mathcal{J}$; and for any $E \in \mathcal{B}(\mathcal{J})$, let $E^\circ, \overline{E}, E^c$, and $\partial E$ denote the interior, closure, complement, and boundary of $E$, respectively. For any $r > 0$ and $y \in \mathbb{R}^d$, let $B_r(y) = \{ x \in \mathbb{R}^d : |x - y| < r \}$. For any measure $\nu$ on $\mathcal{J} \subset \mathbb{R}^d$, let $\text{supp} \nu$ denote the support of $\nu$. Also, denote the set of bounded continuous real-valued functions on a space $E$ by $\mathcal{C}_0(E)$, equipped with the supremum norm, namely $|f|_\infty := \sup \{|f(x)| : x \in E\}$.

Let $\mathcal{M}$ denote the collection of $d \times d$ matrices having nonnegative coefficients, and let $\|m\|$ denote the operator norm, i.e.,

$$||m|| := \sup_{x \in \mathcal{B}^{d-1}} |mx|, \quad m \in \mathcal{M}.$$  

**Allowable and positively regular matrices.** We say that a matrix $m \in \mathcal{M}$ is *allowable* if it has no zero row or column. Moreover, if the coefficients of a given matrix $m \in \mathcal{M}$ are *strictly* positive, then we write $m \succ 0$ and say that $m$ is *positively regular*. With a slight abuse of notation, we write

$$\mathcal{M}^0 = \{ m \in \mathcal{M} : m \succ 0 \}.$$  

As a standing assumption, we shall always assume that there exists an $n \in \mathbb{N}_+$ such that $\mathcal{M} := \inf \{ n \in \mathbb{N}_+ : M_0 \cdots M_1 \succ 0 \}$ is finite a.s.; that is, ultimately, the product $M_n \cdots M_1$ is positively regular with probability one. This assumption will be subsumed in the stronger Hypothesis $(H_1)$, given below (as can be inferred from Hennion (1997), Lemma 3.1 or Buraczewski et al. (2014), Lemma 6.3).

Under this standing assumption, the elements of the vectors $\{ V_n \}$ communicate, leading to a common polynomial decay rate for the exceedance probabilities, regardless of direction, while the pre-factor $C_A := (C/\lambda'(\alpha)) \Sigma_\alpha(A)$ in (1.5), or the constant $K_A$ in (1.6), will be directionally-dependent.

**Non-arithmetic distributions for random matrices.** We will need a generalization of the notion of a non-arithmetic distribution to the setting of random matrices. First consider a more general framework, where $\{(X_n, S_n) : n = 0, 1, \ldots \}$ is a Markov random walk, i.e. $\{(X_n, S_n - S_{n-1})\}$ is a Markov chain with a transition kernel which only depends on the state of the driving chain $\{X_n\}$. The most satisfactory generalization of an arithmetic distribution in this setting is due to Shurenkov (1984). In his formulation, the Markov random walk $\{ (X_n, S_n) : n = 0, 1, \ldots \}$ is *arithmetic* if there exists a $t > 0, \theta \in [0, 2\pi)$, and a “shift-function” $\vartheta$ on $\mathbb{R}$ such that

$$E[\exp \{itS_1 - i\theta + i(\vartheta(X_1) - \vartheta(X_0))\}] = 1,$$

and *non-arithmetic* otherwise. Clearly, if $\{ S_n \}$ denotes the sums of an i.i.d. sequence of random variables (rather than a Markov-dependent sequence), then we may take $\vartheta = 0$, and the above condition is equivalent to the requirement that the support of $S_1$ lies in an arithmetic progression; that is, the distribution of $S_1$ is arithmetic in the classical sense.

This condition is not easily verified in the matrix setting, so it is more natural to adopt a requirement akin to that of Kesten (1973). Namely, let $\Gamma_M$ denote the smallest closed subsemigroup of $\mathcal{M}$ which contains $\text{supp} \mu_M$.

**Definition 2.1.** We say that $\mu_M$ is *non-arithmetic* if the additive group generated by

$$\{ \log \|m\| : m \in \Gamma_M \cap \mathcal{M}^0 \}$$

is dense in $\mathbb{R}$.

It is shown in Buraczewski and Mentemeier (2016), Lemma 2.7, that this condition implies that of Shurenkov (1984). It is also worth observing that, alternatively, we could replace $\log \|m\|$ with the Frobenius eigenvalue of $m$ in Definition 2.1; thus, we see that our definition is, indeed, in agreement with the one given in Kesten (1973).

We are now prepared to introduce our basic assumptions on the distribution function $\mu_M$ of $M$.

**Hypothesis (H$_1$).** $\mu_M$ is non-arithmetic, and $\mu_M \{ m : m \text{ is allowable} \} = 1$.

Since we will employ Markov renewal theory, the appearance of a non-arithmetic assumption is natural. The further requirement that the matrix $M$ is allowable $\mu_M$-a.s. is also standard and appears
in numerous related works in the literature (cf., e.g., Kesten and Spitzer (1984), Hennion (1997), Buraczewski et al. (2014)). In comparison, Kesten (1973) assumes that the rows of \( M \) are nonzero a.s., but does not assume that the columns of \( M \) are also nonzero a.s. (as we assume by requiring that the matrices are “allowable”). This further requirement will guarantee the uniqueness of the invariant measures and functions in Lemma 2.2 below.

We now turn to certain moment conditions which will be imposed on the pair \((M, Q)\). Set

\[
\mathcal{D} = \left\{ \theta \geq 0 : \int_{\mathbb{R}_+} \|m\|^\theta \mu_M(dm) < \infty \right\} = \left\{ \theta \geq 0 : \mathbb{E} \left[ \|M\|^\theta \right] < \infty \right\}.
\]

Let \( m^T \) denote the transpose of \( m \). Then for any \( \theta \in \mathcal{D} \), define:

\[
P_\theta f(x) = \mathbb{E} \left[ |Mx|^\theta f(Mx) \right], \quad f \in \mathcal{C}_b(S_+^{d-1});
\]

\[
P_\theta^r f(x) = \mathbb{E} \left[ |M^r x|^\theta f(M^r x) \right], \quad f \in \mathcal{C}_b(S_+^{d-1});
\]

and

\[
\lambda(\theta) = \lim_{n \to \infty} \left( \mathbb{E} \left[ \|M_n \cdots M_1\|^\theta \right] \right)^{1/n}; \quad \Lambda(\theta) = \log \lambda(\theta).
\]

**Lemma 2.2.** Assume \( \theta \in \mathcal{D} \) and \( \mu_M \{ m : m \text{ is allowable} \} = 1 \). Then \( \lambda(\theta) \) is the spectral radius of \( P_\theta \), and there is a unique probability measure \( l_\theta \) on \( S_+^{d-1} \) and a unique, strictly positive function \( r_\theta \in \mathcal{C}_b(S_+^{d-1}) \) with \( \int_{S_+^{d-1}} r_\theta(x)dl_\theta(x) = 1 \) such that

\[
l_\theta P_\theta = \lambda(\theta)l_\theta, \quad P_\theta r_\theta = \lambda(\theta)r_\theta. \tag{2.2}
\]

Moreover, the function \( r_\theta \) is \( \max\{\theta, 1\} \)-Hölder continuous; thus, in particular, \( r_\theta \) is bounded from above and below by finite positive constants.

Similarly, the spectral radius of \( P_\theta^r \) equals \( \lambda(\theta) \), and there is a unique probability measure \( l_\theta^r \) on \( S_+^{d-1} \) and a unique, strictly positive function \( r_\theta^r \) such that

\[
l_\theta P_\theta^r = \lambda(\theta)l_\theta^r, \quad P_\theta^r r_\theta^r = \lambda(\theta)r_\theta^r, \quad \text{and} \quad \int_{S_+^{d-1}} r_\theta^r(x)dl_\theta^r(x) = 1.
\]

Moreover,

\[
r_\theta(x) = c \int_{S_+^{d-1}} (x,y)^\theta l_\theta(dy) \quad \text{for all } x \in S_+^{d-1}, \tag{2.3}
\]

where \( c = \left( \int_{S_+^{d-1} \times S_+^{d-1}} (x,y)^\theta l_\theta(dx)l_\theta(dy) \right)^{-1} \). Furthermore, (2.3) also holds if \( r_\theta^r \) and \( l_\theta \) are replaced with \( r_\theta \) and \( l_\theta^r \), respectively.

In the above lemma, we have written \( l_\theta P_\theta \) for the application of the adjoint operator \( P_\theta^* \) to the measure \( l_\theta \), i.e. \( l_\theta P_\theta \) is the unique measure satisfying

\[
\int_{S_+^{d-1}} f(x)(l_\theta P_\theta)(dx) = \int_{S_+^{d-1}} (P_\theta f(x)) l_\theta(dx)
\]

for all \( f \in \mathcal{C}_b(S_+^{d-1}) \). The proof of Lemma 2.1 can be found in Buraczewski et al. (2014), Proposition 3.1; see also Guivarc’h and Le Page (2016), Theorem 2.16 for an analogous result in the setting of invertible matrices. For some related results for Harris recurrent Markov chains; see, for example, Nummelin (1984), Ney and Nummelin (1987), or Alsmeyer and Mentemeier (2012).

For any allowable matrix \( m \), now define

\[
\iota(m) := \inf_{x \in S_+^{d-1}} |mx|.
\]
Hypothesis \((H_2)\). There exists an \(\alpha > 0\) such that \(\lambda(\alpha) = 1\), and the following moment conditions hold:

\[
E \left[ \|M\|^n \max \{ \|m\|, \|i(m)\| \} \right] < \infty; \quad \text{and} \quad E[|Q|^\alpha] < \infty.
\]

Once again, \((H_2)\) is quite standard, also in the one-dimensional setting; cf. Goldie (1991). In comparison with our assumptions, Kesten (1973) requires the slightly weaker condition \(E[\|M\|^n \log^+ \|M\|] < \infty\), rather than

\[
E \left[ \|M\|^n \max \{ \|m\|, \|i(m)\| \} \right] < \infty.
\]

However, from our modest strengthening of his condition, we will be able to identify the limit in the Furstenberg-Kesten theorem, as given below (in an extended form) in Lemma 2.3.

The shifted distribution. We shall utilize the constant \(\alpha \) in \((H_2)\) to employ a change of measure. Now in the one-dimensional setting, it is natural to apply this change of measure to the unshifted measure, this sequence will be Markov-dependent in the \(\theta\)-shifted measure. We shall refer to this distribution as the "\(\theta\)-shifted distribution."

It is worth observing that, although \(\{(M_n, Q_n) : n = 1, 2, \ldots\}\) is assumed to be i.i.d. in the unshifted measure, this sequence will be Markov-dependent in the \(\theta\)-shifted measure for any \(\theta > 0\). However, defining

\[
\eta_\theta(E) = \int_E r_\theta(x) \rho(x) dx,
\]

we see that the measure

\[
\hat{\rho}_\theta := \int_{S_+^{d-1}} \rho_\theta(x) dx
\]

is shift-invariant; i.e., the sequence \(\{(M_n, Q_n)\}\) is stationary under \(\hat{\rho}_\theta\); cf. Section 3.1 of Buraczewski et al. (2014). This is an important observation, as it will allow us to apply the results of Hennion (1997) on products of random matrices; cf. Section 4 below. In addition, by Lemma 6.2 of Buraczewski et al. (2014), we have that \(\hat{\rho}_\theta \ll \hat{\rho}_\theta\), for all \(x \in S_+^{d-1}\). We will use this result frequently to infer convergence \(\hat{\rho}_\alpha\)-a.s., for arbitrary \(x \in S_+^{d-1}\), by proving \(\hat{\rho}_\theta\)-a.s. convergence. Furthermore, the finite-dimensional distributions of \(\{(M_i, Q_i) : i = 0, \ldots, n\}\) under each \(\hat{\rho}_\alpha\) are equivalent to their unshifted distributions, since the density function \(p^\theta_n\) is strictly positive.

The Markov random walk. The process \(\{M_n\}\) induces a Markov chain on \(S_+^{d-1}\), defined by setting

\[
X_n = (M_n \cdots M_1 X_0)^{\sim},
\]

for some initial state \(X_0 \in S_+^{d-1}\). This process will play an important role in the sequel. In the \(\theta\)-shifted measure, \(\{X_n\}\) has a unique stationary distribution given by \(\eta_\theta\). If we further define

\[
S_n = \log |M_n \cdots M_1 X_0|, \quad n = 1, 2, \ldots; \quad S_0 = 0;
\]

\[
\text{for some initial state } X_0 \in S_+^{d-1}. \]
then \( \{(X_n, S_n) : n = 0, 1, \ldots\} \) forms a Markov random walk, which we will utilize frequently below due to the fact that, in the \( \alpha \)-shifted measure, \( \{(\tilde{V}_n, \log |V_n| - \log |V_{n-1}|)\} \) closely resembles \( \{(X_n, S_n - S_{n-1})\} \) for large \( n \).

**Probability measures.** We introduce the following conventions to describe conditional probabilities which depend specifically on the initial values of \( X_0 \) and \( V_0 \). Write:

\[
P_v(\cdot) = P(\cdot | V_0 = v), \quad P^\theta_x(X_0 = x) = 1, \quad P^\theta_{x,v}(\cdot) = P^\theta_x(\cdot | V_0 = v),
\]

and use the same notation for expectations. When conditioning on an initial distribution \( V_0 \sim \gamma \), write:

\[
P^\gamma(\cdot) = \int_{\mathbb{R}_+^d \setminus \{0\}} P_v(\cdot) \gamma(dv) = P(\cdot | V_0 \sim \gamma), \quad P^\theta^\gamma(\cdot) = \int_{\mathbb{R}_+^d \setminus \{0\}} P^\theta_v(\cdot) \gamma(dv), \quad P^\gamma_{x,v}(\cdot) = P^\theta_{x,v}(\cdot),
\]

and analogously for expectations. Note that while working in the \( \theta \)-shifted measure, we must specify both \( X_0 \) and \( V_0 \) in these last equations, and we specifically take \( X_0 = \tilde{V}_0 \). The reasoning for this asymmetry comes from the observation that, while the initial state \( X_0 \) does not influence the distribution of \( \{V_n\} \) in the original measure, this initial state does affect the law of \( \{M_n\} \) and hence that of \( \{V_n\} \) in the \( \theta \)-shifted measure and, thus, both of the initial states, \( X_0 \) and \( V_0 \), must be specified in the latter probabilities or expectations. Finally, we will often suppress the dependence on \((x, v)\) and write \( P^\alpha \)-a.s. in place of \( P^\alpha_{x,v} \)-a.s. for all \( x \in \mathbb{S}_+^{d-1}, v \in \mathbb{R}_+^d \setminus \{0\} \). With \( \{(X_n, S_n)\} \) defined as above, using that \( \lambda(\alpha) = 1 \), we can rewrite the change of measure as follows: for all \( n \in \mathbb{N}_+, x \in \mathbb{S}_+^{d-1}, \) and any bounded measurable function \( f : \mathbb{S}_+^{d-1} \times (\mathfrak{N} \times \mathbb{R}_+^d)^n \rightarrow \mathbb{R} \),

\[
r_\alpha(x)E^\alpha_{x,v} \left[ \frac{e^{-nS_n}}{r_\alpha(X_n)} f(X_0, V_0, M_1, Q_1, \ldots, M_n, Q_n) \right] = \mathbb{E} \left[ f(x, v, M_1, Q_1, \ldots, M_n, Q_n) \right]. \tag{2.8}
\]

**Limit theory for the Markov random walk.** Now recall the theorem of Furstenberg and Kesten (1960), which states that, if \( \mathbb{E}[\log\|M_1\|] < \infty \), then

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_n \cdots M_1\| \tag{2.9}
\]

converges to its ergodic average. We will need a refinement of this result, developed in the context of the \( \theta \)-shifted measure, where \( \theta \in \mathfrak{D}^o \). Before stating this result, first recall that under \((H_1), \mathfrak{N} := \inf \{n \in \mathbb{Z}_+ : M_n \cdots M_1 > 0\} < \infty; \) cf. Hennion (1997), Lemma 3.1.

**Lemma 2.3.** Assume that \((H_1)\) is satisfied and let \( \theta \in \mathfrak{D}^o \), and suppose that

\[
\mathbb{E} \left[ \|M\|^\theta \max \{\|\log\|m\|\|, \|\log \hat{m}(m)\|\} \right] < \infty.
\]

Then in the \( \theta \)-shifted measure, we have for all \( x \in \mathbb{S}_+^{d-1} \) that

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log |M_n \cdots M_1 x| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_n \cdots M_1\| = \Lambda'(\theta) = \mathbb{E}^\theta[S_1] \quad \mathbb{P}^\theta \text{-a.s.} \tag{2.10}
\]

and for all \( x, y \in \mathbb{S}_+^{d-1} \),

\[
\lim_{n \rightarrow \infty} \sup \left\{ \left| \frac{1}{n} 1_{\{|n| \leq n\}} \log \langle y, M_n \cdots M_1 x \rangle - \Lambda'(\theta) \right| : x, y \in \mathbb{S}_+^{d-1} \right\} = 0 \quad \mathbb{P}^\theta \text{-a.s.} \tag{2.11}
\]

**Proof.** See Hennion (1997), Theorem 2 (where the uniformity is proved) and Buraczewski et al. (2014), Theorem 6.1 (where the limit is identified in the \( \theta \)-shifted measure).

Finally, when studying the empirical measure conditioned on a large exceedance, it will be helpful to compare with the unconditional behavior of the corresponding \( \alpha \)-shifted Markov random walk. For this purpose, the following lemma is useful.
Lemma 2.4. Assume Hypotheses \((H_1)\) and \((H_2)\) are satisfied, and suppose that the transition kernel of \(\{(X_n, S_n): n = 0, 1, \ldots\}\) follows the \(\alpha\)-shifted measure, \(\mathbb{P}^\alpha\). Then for any measurable function \(g: S_+^{d-1} \times S_+^{d-1} \times \mathbb{R} \to \mathbb{R}\) and any initial state \(X_0 = x \in S_+^{d-1}\),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g(X_k, X_{k-1}, S_k - S_{k-1}) = \mathbb{E}^\alpha \left[ g(X_1, X_0, S_1) \right] \mathbb{P}^\alpha_x \text{-a.s.,}
\]

provided that the expectation on the right-hand side exists.

Proof. This is proved as in the first part of Buraczewski et al. (2014), Theorem 6.1 [employing, in the notation of that article, the function \(f(x, \omega) := g(a_1(\omega) x, x, \log |a_1(\omega) x|)\)].

\[\square\]

2.2 Tail estimates for \(\{V_n\}\)

We now turn to our first main result, where we revisit and extend Kesten’s (1973) theorem, establishing an explicit expression for the constant \(C_A\) in (1.5).

Let \(\pi\) denote the stationary distribution of \(\{V_n\}\), which is given by the law of the random variable

\[
V := \sum_{k=1}^{\infty} M_1 \cdots M_{k-1} Q_{k-1}.
\]

Then it is well known that \(V\) is finite a.s. under the hypotheses of this paper, and hence \(\pi\) exists; cf. Kesten (1973). [The necessary moment hypotheses follow from \((H_2)\), while the negativity of the upper Lyapunov exponent follows from Lemma 2.3 and the convexity of \(\Lambda\), which implies that \(\Lambda'(0) < 0\).]

Now fix a set \(D \subset \mathbb{R}_+^d\), where \(\pi(D) > 0\), and let \(\pi_D\) denote the stationary distribution of \(\{V_n\}\) restricted to \(D\); that is,

\[
\pi_D(E) = \frac{\pi(E \cap D)}{\pi(D)}, \quad E \in \mathcal{B}(\mathbb{R}_+^d);
\]

and let \(\tau\) denote the first return time of \(\{V_n\}\) into \(D\); namely,

\[
\tau = \inf \{n \in \mathbb{N}_+: V_n \in D\}.
\]

Next, let \(\vec{1} = (1, \ldots, 1)^T\), and define

\[
Y_i = \lim_{n \to \infty} \left( M_i^\top \cdots M_1^\top \vec{1} \right)^\sim, \quad n = 1, 2, \ldots.
\]

Note that if \(\theta \in \mathcal{D}\), then the limit on the right-hand side exists \(\mathbb{P}^\theta\text{-a.s.}\), since this product constitutes a backward sequence of an iterated function system and the maps \(\{M_k\}\) act as contractions on \(S_+^{d-1}\); cf. Hennion (1997), Section 3. Moreover, the law of \(Y_i\) is given by

\[
\eta^\theta_\sim(E) := \int_E r^\theta_\sim(x) l^\theta_\sim(dx), \quad E \in \mathcal{B}(S_+^{d-1}),
\]

where \(r^\theta_\sim\) and \(l^\theta_\sim\) are given as in Lemma 2.2 (cf. Guivarc’h and Le Page (2016), Theorem 3.2; Buraczewski et al. (2014), Proposition 3.1).

The condition \((\mathcal{R})\). Next recall that under \((H_1)\), the measure \(\mu_M\) is non-arithmetic and hence \(M_n \cdots M_1\) is positively regular for sufficiently large \(n\) w.p.1, implying that for some positive integer \(k\) and some \(s > 0\),

\[
M_k \cdots M_2 Q_1 \succ s \vec{1} \quad \text{with positive probability. (\mathcal{R})}
\]

Now if \(k > 1\), then it is natural to introduce the \(k\)-step process; namely, for all \(k \in \mathbb{N}_+\), set

\[
\hat{M}_n := M_{kn} \cdots M_{k(n-1)+1} \quad \text{and} \quad \hat{Q}_n = \sum_{i=k(n-1)+1}^{kn} M_i \cdots M_{i+1} Q_i.
\]
and note as a consequence of these definitions that
\[ V_{kn} = \hat{M}_n V_{k(n-1)} + \hat{Q}_n, \quad n = 1, 2, \ldots, \]
where \( \hat{Q}_n - sI > 0 \) with positive probability. It is worth observing here that the stationary distributions of \( \{V_{kn}\} \) and \( \{V_n\} \) are, of course, identical.

Finally, denote by \( \mathcal{C}_0(\mathbb{R}_+^d \setminus \{0\}) \) the set of bounded continuous functions on \( \mathbb{R}_+^d \setminus \{0\} \) which are supported on \( \mathbb{R}_+^d \setminus B_r(0) \), for some \( r > 0 \).

We are now prepared to state our first main result.

**Theorem 2.5.** Assume that Hypotheses \((H_1)\) and \((H_2)\) are satisfied, and suppose that \( D = B_r(0) \cap (\mathbb{R}_+^d \setminus \{0\}) \), where \( r \) is sufficiently large such that \( \pi(D) > 0 \). Then if \( f \in \mathcal{C}_0(\mathbb{R}_+^d \setminus \{0\}) \) and \( k = 1 \) in \((\hat{R})\), we have that
\[ \lim_{u \to \infty} u^\alpha \mathbb{E} \left[ f \left( \frac{V}{u} \right) \right] = \frac{C}{\lambda'(\alpha)} \int_{\mathbb{R}_+^{d-1} \times \mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds, \]
where
\[ C = \int_D r_\alpha(v) \mathbb{E}^\mathbb{Q}_\alpha_{\hat{v}} \left[ \left( |v| + \sum_{i=1}^\infty \frac{\langle Y_i, \hat{Q}_i \rangle}{\langle Y_i, X_i \rangle} |Q_i| |M_i \cdots M_1 \hat{v}| \right)^\alpha \right] 1_{\{\tau = \infty\}} \right] \pi(dv). \]

If \( k > 1 \) in \((\hat{R})\), then the theorem still holds, but with respect to the \( k \)-step chain \( \{V_{kn}\} \) generated by \( \{(\hat{M}_n, \hat{Q}_n)\} \), rather than the 1-step chain \( \{V_n\} \) generated by \( \{(M_i, Q_i)\} \).

If \( \{V_n\} \) is a Harris recurrent chain, then we may always take \( k = 1 \); see Proposition 5.2 below. Moreover, if \( Q > 0 \) with positive probability, then we may again take \( k = 1 \). Thus, the condition \( k = 1 \) is seen to be an exceedingly weak requirement.

More generally, for the \( k \)-step chain, note that the stopping time \( \tau \) in (2.15) is then taken with respect to that chain (rather than the 1-step chain), and the drift factor \( \lambda'(\alpha) \) in (2.14) must be replaced with the drift of the \( k \)-step chain, namely \( k\lambda'(\alpha) \); cf. Remark 5.3 below.

**Remark 2.6.** A more intuitive description of \( C \) is obtained by setting \( Z_n = V_n/|M_n \cdots M_1 V_0| \). Then in Lemma 3.6 below, we will show that \( |Z_n| \to |Z| \) a.s., where \( |Z| \) represents the quantity appearing in (2.15), i.e.,
\[ |Z| = |v| + \sum_{i=1}^\infty \frac{\langle Y_i, \hat{Q}_i \rangle}{\langle Y_i, X_i \rangle} |Q_i| |M_i \cdots M_1 \hat{v}|. \]

Thus, the constant \( C \) is obtained by comparing the growth rate of \( |V_n| \) to the growth rate of \( |M_n \cdots M_1 \hat{v}| \) in the \( \alpha \)-shifted measure, i.e. in a setting where these processes diverge.

It is worth observing that there are other equivalent formulations to (2.14), as follows.

**Remark 2.7.** Let \( \mathcal{L}_\alpha \) be the measure on \( \mathbb{R}_+^d \setminus \{0\} \) defined by the equation
\[ \int_{\mathbb{R}_+^{d-1} \times \mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds = \int_{\mathbb{R}_+^d \setminus \{0\}} f(x) \mathcal{L}_\alpha(dx). \]

Then (2.14) yields the vague convergence (of measures on \( \mathbb{R}_+^d \setminus \{0\} \))
\[ u^{\alpha} \mathbb{P} \left( \frac{V}{u} \in \cdot \right) \xrightarrow{v} \frac{C}{\lambda'(\alpha)} \mathcal{L}_\alpha(\cdot) \quad \text{as } u \to \infty. \]

[Here, \( \mathbb{R}_+^d \setminus \{0\} \) is considered as a subset of the one-point compactification \( (\mathbb{R}_+^d \cup \{\infty\}) \setminus \{0\} \), where sets of the form \( (x, \infty)^d \) are relatively compact. The test functions for vague convergence are those \( f \in \mathcal{C}_0(\mathbb{R}_+^d \setminus \{0\}) \) for which \( \lim_{x \to \infty} f(x) = f(\infty) \) exists.] Now for any measurable set \( A \subset \mathbb{R}_+^d \), which is bounded away from zero and satisfies \( \mathcal{L}_\alpha(\partial A) = 0 \), it follows from the Portmanteau theorem that
\[ \lim_{u \to \infty} u^{\alpha} \mathbb{P}(V \in uA) = \frac{C}{\lambda'(\alpha)} \mathcal{L}_\alpha(A). \]
Thus, in particular, Theorem 2.5 yields an estimate for $P(V \in uA)$ for any open set $A \subset \mathbb{R}^d$ which is bounded away from the origin.

Furthermore, note that for any $t > 0$ and any measurable $E \subset S_{d-1}^+$ with $l_\alpha(\partial E) = 0$, the sets $E^t := \{x \in \mathbb{R}_+^d : |x| > t, x/|x| \in E\}$ are $E_\alpha$-continuous. Hence, for all $E \subset S_{d-1}^+$ with $l_\alpha(\partial E) = 0$,

$$\lim_{u \to \infty} u^{\alpha}P\left(|V| > tu, \frac{V}{|V|} \in E\right) = \frac{C}{\alpha \lambda'(\alpha)} t^{-\alpha} \lambda_\alpha(E). \quad (2.18)$$

Thus, we infer the weak convergence

$$\lim_{u \to \infty} P\left(\frac{V}{|V|} \in \cdot \mid |V| > u\right) \Rightarrow l_\alpha(\cdot). \quad (2.19)$$

In fact, it is easily seen that (2.19) together with $P(|V| > u) \sim (C/\alpha \lambda'(\alpha)) u^{-\alpha}$ also yields (2.14); i.e., these formulations are essentially equivalent. For further information on multivariate regular variation and vague convergence, see Resnick (2004), Section 3.

We conclude this section by comparing our theorem with some related results in the literature. As already noted, in contrast to Kesten (1973), the identification of the constant $C$ is explicit and we have also characterized the directional dependence, whereas Kesten (1973) considers $P(\langle v, V \rangle > u)$ for vectors $v \in \mathbb{R}^d$, which is a special case of (2.14).

In the one-dimensional setting, where (2.1) holds for $(M, Q) \subset (0, \infty) \times [0, \infty)$, it was shown in Collamore and Vidyashankar (2013b) that

$$P(V > u) \sim \frac{C}{\alpha \lambda'(\alpha)} u^{-\alpha} \quad \text{as} \quad u \to \infty, \quad (2.20)$$

where $\lambda(\alpha) = \mathbb{E}[M^\alpha]$ for $\alpha$ chosen such that $\lambda(\alpha) = 1$, and

$$C = \frac{1}{\mathbb{E}_{\pi_d}[\tau]} \mathbb{E}_{\pi_d}^0 \left[ \left(V_0 + \sum_{i=1}^\infty \frac{|Q_i|}{|M_i \cdots M_1|} \right)^\alpha \right]_{\{r = \infty\}}. $$

After an application of Lemma 3.3 (showing that $\pi(D) = (\mathbb{E}_{\pi_d}[\tau])^{-1}$), this constant is easily seen to have the form described in (2.15), since in this simplified setting, $\langle Y_i, \tilde{Q}_i/\langle Y_i, X_i\rangle \rangle = 1$. [In Collamore and Vidyashankar (2013b), the stopping time $\tau$ is taken to be a regeneration time of the process $\{V_n\}$, but the result holds equally well with the return time $\tau$ in place of the regeneration time, utilizing Lemma 3.5 below.] If it is further assumed that $\{M_n\} \subset (0, \infty)$ and $\{Q_n\} \subset (0, \infty)$ are independent, then an alternative characterization of the constant $C$ has also been given by Enriquez et al. (2009) using entirely different techniques.

### 2.3 Extremal estimates for maxima and first passage times

Our next objective is to study the large exceedance probability over a single cycle emanating from, and then returning to, a given set $D \subset \mathbb{R}_+^d \setminus \{0\}$ and, in this way, characterize the distribution of the first passage time $T_u := \inf \{n \in \mathbb{N}_+ : |V_n| > u\}$ and, more generally,

$$T_u^A := \inf \{n \in \mathbb{N}_+ : V_n \in uA\}, \quad \text{where} \quad A \subset \{x \in \mathbb{R}_+^d : |x| > 1\}.$$

[Equivalently, we could assume that $A \subset \mathbb{R}_+^d$ is supported on $\mathbb{R}_+^d \setminus B_r(0)$ for some $r > 0$, and the proofs would still hold with only minor change.] First impose the following additional requirement on the set $A$.

**Definition 2.8.** We say that a set $A \in \mathcal{B}(\mathbb{R}_+^d)$ is a semi-cone if $x \in \partial A \implies \{tx : t > 1\} \subset A$; that is, the ray generated by any point on the boundary of $A$ is entirely contained within the set $A$. 

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Now suppose that \( A \subset \{ x \in \mathbb{R}^d_+ : |x| > 1 \} \) is a semi-cone, and define
\[
d_A(x) = \inf \{ t > 1 : tx \in A \}, \quad x \in \mathbb{S}^{d-1}_+,
\]
and
\[
S_n^A = S_n - \log d_A(X_n), \quad n = 0, 1, 2, \ldots;
\]
\[
r_n^A(x) = r_n(x)(d_A(x))^\alpha, \quad x \in \mathbb{S}^{d-1}_+; \quad \Psi_A = \{ x \in \mathbb{S}^{d-1}_+ : d_A(x) < \infty \}; \tag{2.21}
\]
where \( \{ S_n \} \) is defined as in (2.7).

As a consequence of Kesten’s renewal theorem, it will be shown in Lemma 6.1 below that, if \( \Psi_A = \mathbb{S}^{d-1}_+ \), then
\[
\mathbb{P} \left( M_n \cdots M_1 \tilde{V}_0 \in uA, \text{ for some } n \in \mathbb{N}_+ \middle| V_0 = v \right) \sim r_n(v)D_Au^{-\alpha} \quad \text{as } u \to \infty, \tag{2.22}
\]
where
\[
D_A := \int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}_+} e^{-\alpha s} \frac{1}{r_n(x)} \varrho^A(dx, ds) \tag{2.23}
\]
for a measure \( \varrho^A \) which will be specified below in Section 3.3. Essentially, (2.22) is the ruin estimate for the Markov random walk \( \{ (X_n, S_n^A) : n = 0, 1, \ldots \} \) under the initial state \( X_0 = \tilde{v} \), and \( \varrho^A \) corresponds to the stationary excess distribution for this process. Indeed, if \( A \) is a semi-cone and \( d_A \) is continuous, then it follows immediately from the definitions that, on the left-hand side of (2.22),
\[
M_n \cdots M_1 \tilde{V}_0 \in uA \iff e^{S_n}X_n \in uA \iff S_n^A > \log u - \log d_A(u).
\]
Now if \( \Psi_A \) is strictly contained in \( \mathbb{S}^{d-1}_+ \), then (2.22) will still hold and this defines the constant \( D_A \), although the identification of \( D_A \) is less explicit in this case (i.e., there is no formula equivalent to (2.23)). However, \( D_A \) can nonetheless be interpreted as the ruin constant for the Markov random walk; see Section 6 below.

Finally, let \( C \) be defined as in (2.15) and set
\[
C(v) = r_n(\tilde{v})\mathbb{E}_{\mathbb{S}^d_n}^\alpha \left[ \left( |v| + \sum_{i=1}^\infty \frac{\langle Y_i, \tilde{Q}_i \rangle}{\{ Y_i, X_i \}} \frac{|Q_i|}{\{ M_i \cdots M_i \tilde{v} \}} \right)^\alpha \right] \tag{2.24}
\]

**Theorem 2.9.** Suppose that Hypotheses \((H_1)\) and \((H_2)\) are satisfied and that \( D = B_r(0) \cap \mathbb{R}^d_+ \), where \( r \) is sufficiently large such that \( \pi_r(D) > 0 \). Assume that \( A \in \mathcal{B}(\mathbb{R}^d_+ \setminus B_1(0)) \) is a semi-cone and the function \( d_A \) is continuous. Moreover, assume that \( k = 1 \) in \((\mathcal{R})\). Then for any probability measure \( \gamma \) supported on \( \mathbb{R}^d_+ \setminus \{ 0 \} \),
\[
\lim_{u \to \infty} u^{\alpha}\mathbb{P} \left( T_u^A < \tau \middle| V_0 \sim \gamma \right) = D_A \int_D C(v)\gamma(dv). \tag{2.25}
\]
Furthermore, the sequence \( \{ T_u^A \} \) converges in distribution; more precisely,
\[
\lim_{u \to \infty} \mathbb{P} \left( \frac{T_u^A}{u^{\alpha}} \leq z \middle| V_0 = v \right) = 1 - e^{-K_Az}, \quad z \geq 0, \tag{2.26}
\]
where \( K_A = CD_A \).

We emphasize that the boundary of the set \( A \) is allowed to have an unbounded distance to the origin, so that \( A \) need not intersect every ray emanating from the origin in \( \mathbb{R}^d_+ \). Also, similarly to Theorem 2.5, the assumption that \( k = 1 \) is not necessary if \( \{ V_n \} \) is Harris recurrent, or if \( Q > 0 \) with positive probability.
Remark 2.10. Eq. (2.26) generalizes various known results from extreme value theory relating to the recursive sequence (2.1). For one-dimensional recursions, estimates have previously been given for the distribution of \( \max \{ V_i : 1 \leq i \leq n^{1/\alpha} u \} \) as \( n \to \infty \); cf. de Haan et al. (1989), Theorem 2.1.; Perfekt (1994). In the multidimensional setting, the only result of which we are aware is that of Perfekt (1997), who studies the componentwise maxima, namely the process

\[
\left( \max_{1 \leq i \leq n} V_i, \ldots, \max_{1 \leq i \leq n} V_d \right) \quad \text{as} \quad n \to \infty.
\]

Note that the componentwise maxima need not be achieved simultaneously; hence Perfekt’s results do not coincide with ours. Moreover, in all of these references, additional conditions are assumed which we do not impose here; in particular, in their formulations it must be assumed that \( V_0 \sim \pi \), so that the sequence \( \{ V_n \} \) is stationary.

However, a main contribution of our theorem, beyond its generality, is the explicit identification of the constants involved, namely \( K_A, D_A, \) and \( C \) (which are in fact computable, especially in the one-dimensional case). We emphasize that (2.26) also allows for general sets \( A \) and, thus, it suggests how classical extremal estimates may be naturally extended to multidimensional problems having spatial dependence, replacing maxima with first passage times. Cf. Dembo et al. (1994) for a similar type of estimate in a different multidimensional setting.

Remark 2.11. As a particular application of the previous theorem, we now determine the extremal index of \( \{|V_n|\} \). Integrating with respect to the measure \( \pi \) in (2.26), we obtain

\[
\lim_{u \to \infty} \mathbb{P} \left( \frac{T_A}{u^{\alpha}} \leq z \, \big| \, V_0 \sim \pi \right) = 1 - e^{-K_A z}, \quad z \geq 0.
\]

Now set \( A = \{ x : |x| > 1 \} \). Then it easily follows with \( u = n^{1/\alpha} w \) and \( z = w^{-\alpha} \) that

\[
\lim_{n \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq n} |V_i| \leq n^{1/\alpha} w \, \big| \, V_0 \sim \pi \right) = e^{-K_A w^{-\alpha}}.
\]

Moreover, for this choice of \( A \), we obtain by Theorem 2.5 that

\[
\lim_{n \to \infty} n \mathbb{P} \left( |V| > n^{1/\alpha} w \right) = \frac{C}{\alpha \lambda'(\alpha)} w^{-\alpha}.
\]

Then reasoning as in Leadbetter and Rootzén (1988), Section 2.2, we conclude from (2.27) and (2.28) that the extremal index of \( \{|V_n|\} \) is given by

\[
\Theta = \alpha \lambda'(\alpha) D_A.
\]

For a related result in the one-dimensional setting, see Collamore and Vidyashankar (2013b), Proposition 2.2.

Remark 2.12. If \( \{ V_n \} \) is Harris recurrent, then from the proofs of Theorems 2.5 and 2.9, it can be seen that these results also hold if one replaces \( \tau \) with the first regeneration time of \( \{ V_n \} \), assuming that regeneration has occurred at time 0. However, in this case, the constant in (2.15) takes a slightly different form, namely,

\[
C = \frac{1}{\mathbb{E}_n \mathbb{E}_\nu} \left[ r_\alpha (\tilde{V}_0) \left( |V_0| + \sum_{i=1}^{\infty} \frac{\langle Y_i, \tilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} |M_i \cdots M_1 \tilde{V}_0|} \right]^{\alpha} 1_{\{\tau = \infty\}},
\]

where \( \nu \) comes from the minorization; that is, \( P^k(x, dy) \geq h(x) \nu(dy) \) for a suitable function \( h \) and measure \( \nu \), where \( P \) is the transition kernel of \( \{ V_n \} \).
2.4 The conditional path under a large exceedance and its empirical law

Finally, we consider the path behavior of \( \{V_n\} \) prior to a large exceedance; namely, the conditional law \( \mathbb{P} ( \cdot | T_u^A < \tau ) \) as \( u \to \infty \), where \( \tau \) is the return time to a set \( D = B_r(0) \cap \mathbb{R}_+^d \setminus \{0\} \) with \( \pi(D) > 0 \).

We first recall the classical problem. Suppose that \( S^n_\theta = \xi_1 + \cdots + \xi_n \) is a random walk on \( \mathbb{R} \) with i.i.d. innovations \( \{\xi_i\} \sim G \) and negative mean, and suppose that Cramér’s condition is satisfied; namely,

\[
\int_{\mathbb{R}} e^{\alpha x} G(dx) = 1, \quad \text{for some } \alpha > 0.
\]

Now if one conditions on \( \mathcal{E} := \{S^n_\theta \text{ hits } u \text{ before } 0\} \), then it is well known that the law of \( S^n_\theta \) behaves, as \( u \to \infty \), like its associated random walk (Feller (1971), Section XII.6.(d)); that is, as a random walk whose increments have the \( \alpha \)-shifted distribution

\[
G^\alpha(dx) := e^{\alpha x} G(dx).
\]

Similarly, in ruin theory, it is well known that the likely path to ruin follows a random walk with the \( \alpha \)-shifted distribution; cf. Asmussen (1982). For further information on random walks conditioned to stay nonnegative, we refer to Bertoin and Doney (1994) (who also consider to stay nonnegative \( \alpha \) the \( \alpha \)-shifted distribution). However, we cannot expect so strong a result here, since, as already

The purpose of this section is to make this intuition precise, studying, in the first theorem, the convergence of finite-dimensional distributions of \( \{V_n\} \) under \( \{T_u^A < \tau\} \), and showing that these distributions converge to those of the process \( \{e^{S^n_\theta X \cdot X}\} \) in the \( \alpha \)-shifted measure.

Since \( \{V_n\} \) is an affine recursion, one cannot expect that its behavior will mimic that of \( \{e^{S^n_\theta X \cdot X}\} \) over the entire trajectory. For this reason, we introduce an “initial” level \( \varepsilon_u \) with \( \varepsilon_u = o(u) \) and \( \varepsilon_u \uparrow \infty \) as \( u \to \infty \), and study the trajectory of \( \{V_n\} \) subsequent to its exceedance beyond the level \( \varepsilon_u \), showing that the perturbation by the additive components \( \{Q_n\} \) become asymptotically negligible. Moreover, the asymptotic “overjump” distribution \( V_{T_{\varepsilon_u}} \) can be equated to the asymptotic overjump distribution of \( \{(X_n, S_n)\} \), which we denote by \( g \) and characterize in Subsection 3.3 below.

**Theorem 2.13.** Suppose that Hypotheses (H1) and (H2) are satisfied, and assume that the set \( A \) is a semi-cone and the function \( d_A \) is continuous and bounded on \( S^d_{+1} \). Let \( m \in \mathbb{N}_+ \) and \( g : (\mathbb{R}_+)^{m+1} \to \mathbb{R} \) be \( \theta \)-Hölder continuous for some \( \theta \leq \min \{1, \alpha\} \). Set

\[
I_u = T_{\varepsilon_u}, \quad \text{where } \varepsilon_u = o(u) \quad \text{and} \quad \varepsilon_u \not\to \infty \quad \text{as} \quad u \to \infty.
\]

Then for all \( v \in \mathbb{R}_+^d \),

\[
\lim_{u \to \infty} \mathbb{E}_u \left[ g \left( \frac{V_{I_u}}{|V_{I_u}|}, \ldots, \frac{V_{I_u+m}}{|V_{I_u}|} \right) \right] 1_{T_u^A < \tau} = \int_{S^d_{+1} \times \mathbb{R}_+} \mathbb{E}_x \left[ g(X_0, e^{S_1 X_1}, \ldots, e^{S_m X_m}) \right] \mu(dx, ds).
\]

(The class of \( \theta \)-Hölder continuous functions is a separating class, and thus we deduce, for all \( m \in \mathbb{N}_+ \), the weak convergence

\[
\mathbb{P} \left( \left( \frac{V_{I_u}}{|V_{I_u}|}, \ldots, \frac{V_{I_u+m}}{|V_{I_u}|} \right) \in \cdot \right) \Rightarrow \int_{S^d_{+1} \times \mathbb{R}_+} \mathbb{P}_x^\theta \left( (X_0, e^{S_1 X_1}, \ldots, e^{S_m X_m}) \in \cdot \right) \mu(dx, ds),
\]

for any initial distribution of \( V_0 \).

Note that if we could take \( I_u = 0 \) in the above theorem, then we would obtain an asymptotic description for all paths of finite length, and by the Kolmogorov extension theorem, we could then conclude—as is obtained in Feller (1971) or Bertoin and Doney (1994)—that the conditional path follows the \( \alpha \)-shifted distribution. However, we cannot expect so strong a result here, since, as already
noted, the process \( \{V_n\} \) is not homogeneous and—as a nonlinear renewal process—only resembles the \( \alpha \)-shifted distribution for sufficiently large \( n \) (e.g., for \( n \geq I_u \)).

A stronger version of this theorem—allowing for paths of infinite length—will be proved below in Theorem 7.1. In this general setting, it is natural to consider a scaled process, normalized by a factor \( a^n \) subsequent to time \( I_u \). This normalization is needed, since the distance between \( \{V_n\} \) and \( \{e^{S_n}X_n\} \), in fact, diverges in Theorem 2.13 as \( m \to \infty \). This type of result is consistent with related conditioned limit theorems from large deviation theory; cf. Dembo and Zeitouni (1998), Chapter 7 and references therein.

Nevertheless, in our final result, we consider the complete path between time zero and a large excursion terminating at time \( T_u^A \). Specifically, we prove that the empirical law of the increments \( \{\log |V_n| - \log |V_{n-1}|\} \) along a large excursion has the same limit law, as \( u \to \infty \), as \( \{S_n - S_{n-1}\} \) under \( \mathbb{P}^a \); cf. Lemma 2.4.

**Theorem 2.14.** Suppose that Hypotheses \( (H_1) \) and \( (H_2) \) are satisfied, and assume that the set \( A \) is a semi-cone and the function \( d_A \) is continuous and bounded on \( \mathbb{S}_+^{d-1} \). Then for any \( v \in \mathbb{R}^d \) and any bounded Lipschitz continuous function \( g : \mathbb{R} \to \mathbb{R} \),

\[
\lim_{u \to \infty} \mathbb{E}_v \left[ \frac{1}{T_u^A} \sum_{k=1}^{T_u^A} g \left( \log \left( \frac{|V_k|}{|V_{k-1}|} \right) \right) - \mathbb{E}^a [g(S_1)] \right| T_u^A < \tau \right] = 0. \tag{2.32}
\]

Thus the empirical law of \( \{\log |V_k| - \log |V_{k-1}|\} \) converges weakly, in \( \mathbb{P}_v(\cdot|T_u^A < \tau) \)-probability, to \( \mathbb{P}^a (S_1 \in \cdot) \).

Finally we remark that—under a different formulation from ours—conditioned limit theorems have also been studied recently in Janssen and Segers (2014). In contrast, they consider path behavior conditioned on a large \emph{initial} value, whereas we study \emph{stopped} processes and obtain an entirely distinct characterization in terms of the \( \alpha \)-shifted Markov random walk.

### 2.5 Structure of the paper

The organization of the rest of the paper is as follows. Section 3 contains results about the processes \( \{V_n\} \) and \( \{(X_n,S_n)\} \) which will be needed for the proofs of the main theorems. Specifically, in Subsection 3.1 we quantify recurrence properties of the Markov chain \( \{V_n\} \). Subsection 3.2 contains an essential result characterizing the asymptotic ratio \( |V_n|/e^{S_n} \), which then forms the basis for our application of Meli’s nonlinear Markov renewal theory in Subsection 3.3. In Section 4, we provide a precise description of how the distribution of the post-\( T_v \)-process \( \{V_{T_v+k} : 0 \leq k \leq m \} \) relates to that of \( \{e^{S_k}X_k : 0 \leq k \leq m \} \), for any finite \( m \). This characterization is then used in the proofs of both Theorem 2.5 and Theorem 2.9, given in Sections 5 and 6, respectively. Finally, in Section 7, we prove a stronger version of Theorem 2.13 and provide the proof of Theorem 2.14.

### 3 Background: Markov chain theory and Markov nonlinear renewal theory

In the present section, we present several results from Markov chain theory and nonlinear renewal theory which will be needed for the proofs of the main theorems. After a quick review of the primary results of this section (most importantly, Lemma 3.2, Hypothesis \( (H_3) \), Lemma 3.6, Theorem 3.9, and Theorem 3.14), the reader may wish to proceed to Sections 4-7, where the main results of the paper are proved, referring back to Section 3 as necessary.

#### 3.1 Markov chain theory for \( \{V_n\} \)

Recall that \( \pi \) denotes the stationary distribution of \( \{V_n\} \). In this subsection, we show that the return times to \( \pi \)-positive sets which are neighborhoods of the origin have exponential moments. We further introduce the supplementary Hypothesis \( (H_3) \), which will be used in some proofs in the initial steps, although this hypothesis will ultimately be removed in the proofs of our main theorems.
Lemma 3.1. Assume that \((H_1)\) and \((H_2)\) are satisfied. Then for any \(0 < \theta < \min\{1, \alpha\}\), there exist positive constants \(t < 1\) and \(L < \infty\) such that for \(\mathbb{D}^\dagger := \{v \in \mathbb{R}_+^d : |v| \leq L\}\),

\[
\mathbb{E}\left[|V_n|^\theta r_\theta(\tilde{V}_n) \mid \mathcal{F}_{n-1}\right] \leq t|V_{n-1}|^\theta r_\theta(\tilde{V}_{n-1}), \quad \text{for all } V_{n-1} \in \mathbb{R}_+^d \setminus \mathbb{D}^\dagger. \tag{3.1}
\]

In particular, for \(\tau^\dagger := \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}^\dagger\}\), there exists a finite constant \(B\) such that

\[
\mathbb{E}\left[|V_n|^\theta 1_{\{\tau^\dagger > n\}} | V_0 = v \right] \leq B t^n |v|^\theta, \quad \text{for all } v \in \mathbb{R}_+^d \setminus \mathbb{D}^\dagger. \tag{3.2}
\]

Note that (3.1) can be viewed as an extension of a standard drift condition from Markov chain theory, typically used to ensure that the chain is geometrically recurrent under the additional assumption of \(\psi\)-irreducibility (cf. Nummelin (1984), Chapter 5, or Meyn and Tweedie (1993), Section 14.2).

**Proof.** From Lemma 2.2 (specifically, (2.3) with \((r_\theta, l^*_\theta)\) in place of \((r^*_\theta, l_\theta)\)), we have that for some constant \(c \in (0, \infty)\),

\[
|V_n|^\theta r_\theta(\tilde{V}_n) = c|V_n|^\theta \int_{\mathbb{S}_+^{d-1}} \langle y, \tilde{V}_n \rangle^\theta l^*_\theta(dy) = c \int_{\mathbb{S}_+^{d-1}} \langle y, V_n \rangle^\theta l^*_\theta(dy),\tag{3.3}
\]

where \(V_n = M_n V_{n-1} + Q_n\). Then it follows by subadditivity that for any \(\theta \in (0,1)\),

\[
\mathbb{E}\left[|V_n|^\theta r_\theta(\tilde{V}_n) \mid \mathcal{F}_{n-1}\right] \leq c \mathbb{E}\left[\int_{\mathbb{S}_+^{d-1}} \left(\langle y, M_n V_{n-1} \rangle^\theta + \langle y, Q_n \rangle^\theta\right) l^*_\theta(dy) \mid V_{n-1}\right]. \tag{3.4}
\]

To identify the quantity on the right-hand side, first apply (3.3) once more to obtain that

\[
c \mathbb{E}\left[\int_{\mathbb{S}_+^{d-1}} \langle y, M_n V_{n-1} \rangle^\theta l^*_\theta(dy) \mid V_{n-1}\right] = \mathbb{E}\left[|M_n V_{n-1}|^\theta r_\theta((M_n V_{n-1})^\sim) \mid V_{n-1}\right]
\]

\[= |V_{n-1}|^\theta \int |m(V_{n-1})^\sim|^\theta r_\theta(m(V_{n-1})^\sim) \mu_M(dm)
\]

\[= |V_{n-1}|^\theta \cdot P_\theta r_\theta(V_{n-1}) = |V_{n-1}|^\theta \cdot \lambda(\theta) r_\theta(V_{n-1}),\tag{3.5}
\]

where the operator \(P_\theta\) was defined just prior to Lemma 2.2, and, by that lemma, \(P_\theta\) has eigenvalue \(\lambda(\theta)\) and the right-invariant function \(r_\theta\). Moreover, since \(l^*_\theta\) is a probability measure and \(Q_n\) is independent of \(V_{n-1}\),

\[
\mathbb{E}\left[\int_{\mathbb{S}_+^{d-1}} \langle y, Q_n \rangle^\theta l^*_\theta(dy) \mid V_{n-1}\right] \leq \mathbb{E}\left[|Q_n|^\theta\right]. \tag{3.6}
\]

Then substituting (3.5) and (3.6) in (3.4) yields

\[
\mathbb{E}\left[|V_n|^\theta r_\theta(\tilde{V}_n) \mid \mathcal{F}_{n-1}\right] \leq \lambda(\theta)|V_{n-1}|^\theta r_\theta(\tilde{V}_{n-1}) + c \mathbb{E}\left[|Q_n|^\theta\right]. \tag{3.7}
\]

Now by Hypothesis \((H_2)\), \(\mathbb{E}\left[|Q_1|^\theta\right] < \infty\) for \(0 \leq \theta \leq \alpha\). Moreover, \(0 < \theta < \min\{\alpha, 1\} \implies \lambda(\theta) < 1\). Choosing \(t \in (\lambda(\theta), 1)\) and then choosing \(L < \infty\) sufficiently large, we conclude by (3.7) that (3.1) is satisfied.

To obtain (3.2), iterate (3.1) to deduce that

\[
\mathbb{E}\left[|V_n|^\theta r_\theta(\tilde{V}_n) 1_{\{\tau^\dagger > n\}} \mid V_0 \right] \leq t^n |V_0|^\theta r_\theta(\tilde{V}_0),
\]

and use that the function \(r_\theta\) is bounded from above and below, by Lemma 2.2. \(\square\)

Next, recall that \(\pi\) denotes the stationary measure of \(\{V_n\}\).
Lemma 3.2. Suppose \((H_1)\) and \((H_2)\) are satisfied, and let \(\mathbb{D} = B_v(0) \cap \mathbb{R}^d_+ \setminus \{0\}\) for some \(r > 0\) such that \(\pi(\mathbb{D}) > 0\). Let \(\mathbb{D}^\dagger = \{v \in \mathbb{R}^d_+ : |v| \leq L\}\), where \(L\) is chosen such that (3.2) is satisfied and \(\mathbb{D}^\dagger \supset \mathbb{D}\). Then there exist constants \(t \in (0, 1)\) and \(B < \infty\) such that, for \(\tau = \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}\}\),

\[
\sup_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(\tau > n \mid V_0 = v\right) \leq B t^n, \quad \text{for all } n \in \mathbb{N}_+. \tag{3.8}
\]

**Proof.** As in the previous lemma, let \(\tau^\dagger\) denote the first return time of \(\mathbb{D}^\dagger\). Then (3.2) gives for all \(v \in \mathbb{D}^\dagger\) and \(n \geq 1\) that

\[
\mathbb{E}\left[|V_n|^\theta 1_{\{\tau^\dagger > n\}} \mid V_0 = v\right] = \mathbb{E}\left[\mathbb{E}\left[|V_n|^\theta 1_{\{\tau^\dagger > n\}} \mid V_1\right] 1_{\{V_1 \notin \mathbb{D}^\dagger\}} \mid V_0 = v\right] \\
\leq B_t t^{n-1} \mathbb{E}\left[|V_1|^\theta 1_{\{V_1 \notin \mathbb{D}^\dagger\}} \mid V_0 = v\right] \leq B_t t^{n-1} \left(\mathbb{E}\|M\|^\theta L^\theta + \mathbb{E}\|Q\|^\theta\right).
\]

Since \(|V_n| > L\) on \(\{\tau^\dagger > n\}\), it follows that for some finite constant \(B_2\),

\[
\sup_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(\tau^\dagger > n \mid V_0 = v\right) \leq B_2 t^n. \tag{3.9}
\]

Below we shall prove that for some constant \(s > 0\) and \(k \in \mathbb{N}_+\),

\[
\sup_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(\tau > k \mid V_0 = v\right) \leq (1 - s). \tag{3.10}
\]

Now assume that (3.9) and (3.10) hold. Without loss of generality, we may further assume that \(k = 1\), for if \(k > 1\), then we may consider the \(k\)-step chain \(\{V_{nk} : n = 0, 1, \ldots\}\) instead of \(\{V_n\}\), and note that if \(\{V_{kn}\}\) returns to \(\mathbb{D}\) at a geometric rate, then so does \(\{V_n\}\).

Thus, assume that \(k = 1\) in (3.10), and observe from (3.9) that \(\tau^\dagger\) has exponential moments; in particular, there exists a constant \(\varepsilon > 0\) such that

\[
\sup_{v \in \mathbb{D}^\dagger} \mathbb{E}_v[e^{\varepsilon \tau^\dagger}] < \frac{1}{1 - s}.
\]

Let \(\tau^\dagger_1, \tau^\dagger_2, \ldots\) denote the successive returns of \(\{V_n\}\) to \(\mathbb{D}^\dagger\). Now if \(N\) denotes the random number of returns to \(\mathbb{D}^\dagger\) prior to time \(\tau\), then

\[
\sup_{v \in \mathbb{D}^\dagger} \mathbb{E}_v[e^{\varepsilon \tau^\dagger}] \leq \sum_j \sup_{v \in \mathbb{D}^\dagger} \mathbb{E}_v\left[e^{\varepsilon (\tau^\dagger_1 + \cdots + \tau^\dagger_j + 1)}\right] \mathbb{P}(N = j) \leq e^\varepsilon \sum_j \left(\sup_{v \in \mathbb{D}^\dagger} \mathbb{E}_v\left[e^{\varepsilon \tau^\dagger}\right]\right)^j (1 - s)^j < \infty,
\]

and (3.8) follows.

To establish (3.10), we use Proposition 4.3.1 of Buraczewski, Damek, and Mikosch (2016), which gives a precise description of \(\text{supp} \pi\). Namely, there exists a set \(\mathcal{S}\) with \(\overline{\mathcal{S}} = \text{supp} \pi\). Moreover, for each \(v_0 \in \mathcal{S}\), there exists \(l \in \mathbb{N}_+\) and \(m_1, \ldots, m_l \in \text{supp} \mu_M, q_1, \ldots, q_l \in \text{supp} \mu_Q\) such that

\[
h : v \mapsto m_l \cdots m_1 v + \sum_{i=1}^l m_i \cdots m_{i+1} q_i
\]

is a contraction on \(\mathbb{R}^d_+\) with \(v_0\) as the unique fixed point. Hence, using that \(\mathbb{D}^\dagger\) is compact, we obtain that for any \(\delta > 0\), there exists \(j \in \mathbb{N}_+\) such that \(|h^j(v) - v_0| < \delta/2\) for all \(v \in \mathbb{D}^\dagger\). Then, from continuity and the definition of the support, we conclude that

\[
\inf_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(|V_{lj} - v_0| < \delta \mid V_0 = v\right) > 0. \tag{3.11}
\]

Since \(\mathbb{D}\) is open and \(\pi(\mathbb{D}) > 0\), and hence \(\mathbb{D} \cap \text{supp} \pi \neq \emptyset\), it follows that \(\mathbb{D} \cap \mathcal{S} \neq \emptyset\) as well. Now let \(v_0 \in \mathbb{D} \cap \mathcal{S}\) and choose \(\delta > 0\) such that \(B_\delta(v_0) \in \mathbb{D}\). Then (3.10) follows from (3.11) with \(k = lj\). □
From the previous result we infer that \( \{V_n\} \) returns to \( \mathbb{D} \) at a geometric rate, starting from a state in \( \mathbb{D}^1 \supset \mathbb{D} \). In the next result, we calculate the expected return time, now starting from the stationary distribution restricted to \( \mathbb{D} \), and provide a law of large numbers for the return times. First let \( \kappa_0 = 0 \) and

\[
\kappa_i = \inf \{ n > \kappa_{i-1} : V_n \in \mathbb{D} \}, \quad i = 1, 2, \ldots;
\]

and let \( \tau_i := \kappa_i - \kappa_{i-1} \) denote the inter-return times, \( i = 1, 2, \ldots \). Set \( N_D(n) = \sum_{k=1}^{n} \mathbb{1}_D(V_k) \).

**Lemma 3.3.** Suppose Hypotheses (H1) and (H2) are satisfied and \( \pi(\mathbb{D}) > 0 \). Then for \( \pi \)-a.e. \( v \in \mathbb{R}^d_+ \),

\[
\lim_{i \to \infty} \frac{\kappa_i}{i} = \lim_{n \to \infty} \left( \frac{N_D(n)}{n} \right)^{-1} = \frac{1}{\pi(\mathbb{D})} \quad \mathbb{P}_v\text{-a.s.} \tag{3.12}
\]

and \( \pi(\cdot) := \pi(\cdot)/\pi(\mathbb{D}) \) is invariant with respect to the process \( \{V_{\kappa_i} : i = 0, 1, \ldots \} \). Moreover, as an alternative representation to (3.12), we also have that

\[
\lim_{i \to \infty} \frac{\kappa_i}{i} = \mathbb{E}_{\pi_D}[\tau] \quad \mathbb{P}_{\pi_D}\text{-a.s.} \tag{3.13}
\]

[On the right-hand sides of (3.12) and (3.13), we recall that \( \mathbb{P}_v\text{-a.s.}, \mathbb{P}_{\pi_D}\text{-a.s.} \) mean that these results hold a.s. provided that the initial value is \( V_0 = v \), or the initial distribution is \( V_0 \sim \pi_D \), respectively.]

**Proof.** If \( V_0 \sim \pi \), then the sequence \( \{V_n\} \) is stationary. Moreover, since each \( V_n \) is a function of the ergodic sequence \( \{(M_n, Q_n)\} \), it follows from Proposition 6.31 of Breiman (1968) that \( \{V_n\} \) is ergodic. Hence, by Birkhoff’s ergodic theorem, we have for any \( \pi \)-integrable measurable function \( h \) that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} h(V_k) = \int_{\mathbb{R}^d_+} h(x) \pi(dx) \quad \mathbb{P}_v\text{-a.s.}, \tag{3.14}
\]

for \( \pi \)-a.e. \( v \in \mathbb{R}^d_+ \setminus \{0\} \). Setting \( h = \mathbb{1}_D \) then yields

\[
\lim_{n \to \infty} \frac{N_D(n)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{1}_D(V_k) = \pi(\mathbb{D}) \quad \mathbb{P}_v\text{-a.s.} \tag{3.15}
\]

As \( \kappa_0, \kappa_1, \ldots \) denote the successive return times to \( \mathbb{D} \), it follows by definition that \( N_D(\kappa_1) = 1 \), \( N_D(\kappa_2) = 2 \), and so on. Thus

\[
1 = \frac{N_D(\kappa_i)}{\kappa_i} \cdot \frac{\kappa_i}{i}.
\]

Noting that \( \kappa_{i+1} \geq i \) and applying (3.15) along the subsequence \( \{\kappa_i\} \), we then obtain that

\[
\lim_{i \to \infty} \frac{\kappa_i}{i} = \lim_{n \to \infty} \left( \frac{N_D(n)}{n} \right)^{-1} = \frac{1}{\pi(\mathbb{D})} \quad \mathbb{P}_v\text{-a.s.}, \tag{3.16}
\]

which is (3.12). In particular, this proves the recurrence of \( \mathbb{D} \), and hence we may apply Proposition VII.3.4 of Asmussen (2003) to infer that \( \pi_D \) is an invariant probability measure with respect to the process \( \{V_{\kappa_i}\} \).

Finally suppose that \( V_0 \sim \pi_D \). Then \( \tau_i \equiv \kappa_i - \kappa_{i-1} \) is stationary and by the ergodic theorem,

\[
\lim_{i \to \infty} \frac{\kappa_i}{i} = \mathbb{E}_{\pi_D}[\tau] \quad \mathbb{P}_{\pi_D}\text{-a.s.}, \tag{3.17}
\]

using that the left-hand side of this equation converges a.s. to a deterministic limit, by (3.16). \( \square \)

Now let \( P \) denote the transition kernel of \( \{V_n\} \). We conclude this section with two results which hold under the following additional Hypothesis (H3).

**Hypothesis (H3).** (i) There exists a \( \pi \)-positive set \( F \) such that, for each \( v \in F \), \( P(v, \cdot) \) has an absolutely continuous component with respect to some \( \sigma \)-finite non-null measure \( \Phi \).

(ii) \( (\text{supp } \pi)^0 \neq \emptyset \).
Lemma 3.4. Assume that (H1), (H2), and (H3) are satisfied. Then \( \{V_n\} \) is an aperiodic, positive Harris chain on \( \mathbb{R}^d_+ \). Moreover, \( \{V_n\} \) is \( \psi \)-irreducible, regular, and geometrically recurrent.

**Proof.** Under (H3), it follows from Alsmeyer (2003), Theorem 2.1 (b) and Theorem 2.2 (b) that \( \{V_n\} \) is an aperiodic, positive Harris chain on \( \mathbb{R}^d_+ \). Hence by Meyn and Tweedie (1993), Theorem 13.0.1,  
\[ \sup_{E \in \mathcal{B}(\mathbb{R}^d_+)} |P^n(x, E) - \pi(E)| \to 0 \quad \text{as} \quad n \to \infty, \]
for all \( x \in \mathbb{R}^d_+ \). This implies, in particular, that the chain is \( \pi \)-irreducible (and hence \( \psi \)-irreducible for some maximal irreducibility measure \( \psi \)).

Since \( \text{supp } \pi^0 \neq \emptyset \), we also have that every compact set with positive invariant measure is petite (Nummelin and Tuominen (1982), Remark 2.7 or Meyn and Tweedie (1993), Proposition 6.2.8). Let \( L \) be chosen sufficiently large such that \( D^L := \{|v| \leq L\} \) has positive invariant measure and (3.1) holds. By Lemma 3.2, we have \( \sup_{v \in D^L} \mathbb{E} [\tau^L | V_0 = v] < \infty \) for the return time \( \tau^L \) of \( D^L \). Then by Meyn and Tweedie (1993), Theorem 11.3.15, we conclude that \( \{V_n\} \) is regular. Moreover, from the above calculation given in the proof of Lemma 3.2,
\[ \sup_{v \in D^L} \mathbb{E} [\varepsilon^{\tau^L}] < \infty, \quad \text{some } \varepsilon > 0, \]
and hence \( \{V_n\} \) is geometrically recurrent. \( \Box \)

Using the \( \psi \)-irreducibility from the previous lemma, we may observe the following useful result, connecting the stationary distribution of \( \{V_n\} \) to its average behavior over a given cycle emanating from a \( \pi \)-positive set \( D \) with initial measure \( \pi_D(\cdot) := \pi(\cdot \cap D)/\pi(D) \).

**Lemma 3.5.** Suppose that (H1) – (H3) are satisfied, and let \( D \subset \mathbb{R}^d_+ \setminus \{0\} \) be chosen such that \( \pi(D) > 0 \). Let \( \tau := \inf \{n \in \mathbb{N}_+ : V_n \in D \} \) denote the first return time of \( D \). Then for any \( \pi \)-integrable function \( h \),
\[ \int h(v) \pi(dv) = \mathbb{E} [h(V)] = \frac{1}{\mathbb{E}_{\pi_D}[\tau]} \mathbb{E}_{\pi_D}\left[ \sum_{i=0}^{\tau-1} h(V_i) \right], \quad (3.18) \]

**Proof.** See Nummelin (1984), Proposition 5.9 and the discussion just prior to Corollary 5.3. For a closely related result, also see the proof of Theorem 2.1 in Collamore et al. (2014). \( \Box \)

### 3.2 Quantifying the discrepancy between \( \{V_n\} \) and \( \{e^{S_n}X_n\} \)

The objective of this subsection is to precisely quantify the discrepancy between \( \{V_n\} \) and \( \{e^{S_n}X_n\} \), which, in essence, will later be shown to determine the constant \( C \) in Theorem 2.5. To this end, let
\[ Z_n := \frac{V_n}{|M_n \cdots M_1 X_0|} = \frac{V_n}{e^{S_n}}, \quad n \in \mathbb{N} \quad (3.19) \]
and
\[ Z_n^{(0)} := \frac{V_n - M_n \cdots M_1 V_0}{|M_n \cdots M_1 X_0|} = \frac{\sum_{i=1}^n M_n \cdots M_{i+1} Q_i}{|M_n \cdots M_1 X_0|}, \quad n \in \mathbb{N}. \]

Also introduce the shorthand notation
\[ \Pi_n := M_n \cdots M_1, \quad \text{and} \quad \Pi_n^a := M_n \cdots M_1. \]

The most important properties of \( \{Z_n\} \), for our purposes, are summarized in the following.

**Lemma 3.6.** Assume (H1) and (H2). Then:
(i) \( \sup_{n \in \mathbb{N}} |Z_n| < \infty \) \( \mathbb{P}^a \)-a.s. and \( \sup_{n \in \mathbb{N}} |Z_n^{(0)}| < \infty \) \( \mathbb{P}^a \)-a.s.
(ii) Let \( v \in \mathbb{R}^d_+ \setminus \{0\} \). Then in \( \mathbb{P}_\delta^\alpha \)-measure, the sequence \( \{Z_n\} \) converges in law to a random variable \( Z \), and \( |Z_n| \to |Z| \) a.s., where

\[
|Z| = |v| + \sum_{i=1}^{\infty} \frac{\langle Y_i, \hat{Q}_i \rangle}{\langle Y_i, X_i \rangle} |Q_i| \quad \mathbb{P}_\delta^\alpha\text{-a.s.} \tag{3.20}
\]

Moreover, \( |Z| \) is strictly positive and finite \( \mathbb{P}_\delta^\alpha\text{-a.s.} \). Similarly, we have

\[
\lim_{n \to \infty} |Z^{(0)}_n| = \sum_{i=1}^{\infty} \frac{\langle Y_i, \hat{Q}_i \rangle}{\langle Y_i, X_i \rangle} |Q_i| \quad \mathbb{P}_\delta^\alpha\text{-a.s.} \tag{3.21}
\]

(iii) Let \( F \subseteq \mathbb{R}^d_+ \setminus \{0\} \) be a bounded set and let \( \tau' \) be any \( \{\mathcal{F}_n\} \)-stopping time such that

\[
\sup_{v \in F} \mathbb{P} \left( \tau' > k | V_0 = v \right) \leq B t^k, \quad \text{for all } k \in \mathbb{N}, \tag{3.22}
\]

for some finite constant \( B \) and \( t \in (0, 1) \). Then for any \( v \in \mathbb{R}^d_+ \setminus \{0\} \),

\[
\sup_{v \in F} \mathbb{E}_\delta^\alpha \left[ \sup_{n \in \mathbb{N}} |Z_n|^\alpha 1_{\{\tau' \geq n\}} \right] < \infty \quad \text{and} \quad \sup_{v \in F} \mathbb{E}_\delta^\alpha \left[ |Z|^\alpha 1_{\{\tau' = \infty\}} \right] < \infty. \tag{3.23}
\]

(iv) For \( v \in \mathbb{R}^d_+ \setminus \{0\} \), we have the \( L^1 \)-convergence

\[
\lim_{n \to \infty} \mathbb{E}_\delta^\alpha \left[ \left| |Z_n|^\alpha 1_{\{\tau' \geq n\}} - |Z|^\alpha 1_{\{\tau' = \infty\}} \right| \right] = 0. \tag{3.24}
\]

Note that by Lemma 3.2, the condition in (iii) holds, in particular, for \( \tau' = \tau := \inf \{n \in \mathbb{N}_+ : V_n \in \mathcal{D}\} \), namely the return time of \( V_n \) into the set \( \mathcal{D} \).

**Proof.** For any vector \( x \in \mathbb{R}^d \), let \( x^{(i)} = (e_i, x) \) denote the \( i \)-th component of \( x \), and set \( \tilde{\mathcal{F}} = (1, \ldots, 1)^T \).

Also, except in part (iii), fix \( V_0 = v \) throughout the proof.

First recall that any \( \mathbb{P}_{\Sigma, \alpha}^\alpha \) is absolutely continuous with respect to \( \hat{\mathbb{P}}^\alpha \) (Buraczewski et al. (2014), Lemma 6.2), and hence the convergence of \( \{Z_n\} \) in law, or the convergence of \( \{\|Z_n\|\} \) \( \hat{\mathbb{P}}^\alpha \)-a.s., implies the respective convergence under \( \mathbb{P}_{\Sigma, \alpha}^\alpha \). Thus, it is sufficient to prove the convergence results in part (i) and (ii) with respect to the measure \( \hat{\mathbb{P}}^\alpha \), under which the sequence \( \{(M_n, Q_n) : n = 1, 2, \ldots\} \) is stationary: cf. the discussion in Section 2 above. This will allow us to apply the results of Hennion (1997).

(i) Suppose \( m \in \mathfrak{M} \), and let \( x_m \) be chosen such that \( \|m\| = |m x_m| \). Since \( m \) is nonnegative, an elementary argument shows that \( x_m \) can, in fact, be chosen such that \( x_m^{(i)} \geq 0 \) for all \( i \). Then for any \( x \in \mathbb{S}_{\delta, 1}^{d-1} \),

\[
|m x| \geq \left( \min_j x^{(j)} \right) \left| m \right| \geq \left( \min_j x^{(j)} \right) |m x_m| = \left( \min_j x^{(j)} \right) \|m\|.
\]

Thus

\[
\frac{\|m\|}{|m x|} \leq \frac{1}{\min_j x^{(j)}}, \quad \text{for all } x \in \mathbb{S}_{\delta, 1}^{d-1} \text{ and all } m \in \mathfrak{M}.
\]

Recall the stopping time \( N := \inf \{n \in \mathbb{Z}_+ : \Pi_n > 0\} \), which is finite \( \hat{\mathbb{P}}^\alpha \)-a.s. by \((H_1)\). [Since \( \mu_M \) is equivalent to \( \hat{\mathbb{P}}^\alpha (M_1 \in \cdot) \), \((H_1)\) holds equally well for \( \hat{\mathbb{P}}^\alpha (M_1 \in \cdot) \). Then Lemma 3.1 of Hennion (1997) yields the finiteness of \( \mathfrak{N} \).] Identifying \( Q_0 := V_0 = v \), we obtain

\[
|Z_n| \leq \sum_{i=0}^{n} \frac{\|\Pi_{i+1}^n Q_i\|}{\|\Pi_n X_0\|} \leq \sum_{i=0}^{n} \frac{\|\Pi_{i+1}^n\| |Q_i|}{\|\Pi_{i+1} X_i\| |\Pi_n X_0|} \leq \sum_{i=0}^{n} \frac{\|\Pi_{i+1}^n\| |Q_i|}{\|\Pi_{i+1} X_i\| |\Pi_n X_0|} \leq \frac{1}{\min_j x^{(j)} |\Pi_n X_0|} \tag{3.25}
\]
By Buraczewski et al. (2014), Lemma 6.3,

\[ C_i(x) := \inf_{n \in \mathbb{N}} \frac{\|\Pi_i^n x\|}{\|\Pi_i^n\|} > 0 \quad \hat{\mathbb{P}}^\alpha \text{-a.s.,} \]

for all \( x \in S_{+}^{d-1} \). Also observe that

\[ X_i^{(j)} = \frac{(\Pi_i X_0)^{(j)}}{\|\Pi_i X_0\|}, \tag{3.26} \]

which implies that \( X_i^{(j)} |\Pi_i X_0| = (\Pi_i X_0)^{(j)} = \langle e_j, \Pi_i X_0 \rangle \). This identifies the denominator in the second sum of (3.25), and shows that this denominator is positive for \( i \geq \mathfrak{R} \). Hence

\[
\sup_{n \in \mathbb{N}} |Z_n| \leq \sum_{k=0}^{\mathfrak{R}} \frac{|Q_j|}{C_i(X_i)|\Pi_i X_0|} + \sum_{i=\mathfrak{R}}^\infty \min_j \langle e_j, \Pi_i X_0 \rangle
\]

\[
\leq \sum_{k=0}^{\mathfrak{R}} \frac{|Q_j|}{C_i(X_i)|\Pi_i X_0|} + \sum_{i=\mathfrak{R}}^\infty \sum_{j=1}^d \langle e_j, \Pi_i X_0 \rangle. \tag{3.27}
\]

Since \( \mathfrak{R} < \infty \) \( \hat{\mathbb{P}}^\alpha \)-a.s., it suffices to focus on the second sum. By Lemma 2.3,

\[ \lim_{n \to \infty} \sup \left\{ \frac{1}{n} 1_{\{\mathfrak{R} \leq n\}} \log \langle y, \Pi_n x \rangle - \Lambda'(\alpha) : x, y \in S_{+}^{d-1} \right\} = 0 \quad \hat{\mathbb{P}}^\alpha \text{-a.s.} \tag{3.28} \]

Furthermore, by a Borel-Cantelli argument, \( \hat{\mathbb{P}}^\alpha (\log |Q_i| > \delta i \text{ i.o.}) = 0 \), for all \( \delta > 0 \). Thus, given \( \varepsilon \in (0, \Lambda'(\alpha)) \), there exists a finite integer \( k_0 \) such that, for all \( i \geq k_0 \) and all \( j \in \{1, \ldots, d\} \),

\[ \log |Q_i| - \log \langle e_j, \Pi_i X_0 \rangle \leq - \left( \Lambda'(\alpha) - \varepsilon \right) i \quad \hat{\mathbb{P}}^\alpha \text{-a.s.} \tag{3.29} \]

Since (3.29) holds uniformly in \( j \), substituting (3.29) into (3.27) establishes part (i) of the lemma, where we also use that \( |Z_n^{(0)}| \leq |Z_n| \) for all \( n \in \mathbb{N} \).

(ii) Following Hennion (1997), let \( \varrho(\Pi^n_k) \) denote the spectral radius of \( \Pi^n_k \), and let \( R^n_k \) and \( L^n_k \) denote the right and left eigenvectors corresponding to the maximal eigenvalue in modulus; that is,

\[ (\Pi^n_k) R^n_k = \varrho(\Pi^n_k) R^n_k \quad \text{and} \quad (\Pi^n_k)^T L^n_k = \varrho(\Pi^n_k) L^n_k, \quad 1 \leq k \leq n. \]

Note that the Perron-Frobenius theorem assures that \( R^n_k \) and \( L^n_k \) have nonnegative entries. We further assume the following normalization:

\[ |L^n_k| = 1, \quad \langle L^n_k, R^n_k \rangle = 1, \quad 1 \leq k \leq n. \]

Now let \( \{Y_i\} \) be defined as in (2.13). Then we will show that

\[ \lim_{n \to \infty} \left| \langle e_j, Z_n \rangle - \langle e_j, \tilde{R}^1_n \rangle \sum_{i=0}^n \frac{\langle Y_i, \tilde{Q}_i \rangle |Q_i|}{\langle Y_i, X_i \rangle |\Pi_i X_0|} \right| = 0 \quad \hat{\mathbb{P}}^\alpha \text{-a.s.,} \tag{3.30} \]

for all \( 1 \leq j \leq d \). The sequence \( \{\tilde{R}^1_n\} \) converges in distribution as \( n \to \infty \) (Hennion (1997), Theorem 1 (ii) (b)); hence we obtain the convergence, in distribution, of \( \{Z_n\} \) to

\[ Z := \lim_{n \to \infty} \tilde{R}^1_n \cdot \lim_{n \to \infty} \sum_{i=0}^n \frac{\langle Y_i, \tilde{Q}_i \rangle |Q_i|}{\langle Y_i, X_i \rangle |\Pi_i X_0|}. \]

Moreover, since \( |\tilde{R}^1_n| = 1 \), (3.30) yields (3.20), i.e. \( \lim_{n \to \infty} |Z_n| = |Z| \) \( \hat{\mathbb{P}}^\alpha \)-a.s. In the same way, (3.21) is obtained by setting \( Q_0 = 0 \).
To establish (3.30), first recall that (with the identification $Q_0 := V_0 = v$) we have that

\[ Z_n = \sum_{i=0}^{n} \frac{\Pi_i^n Q_i}{|\Pi_n X_0|} \]

and observe that

\[
\left| \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{\langle e_j, \Pi_i^n Q_i \rangle}{|\Pi_n X_0|} \right| \leq \sum_{i=\lfloor n/2 \rfloor + 1}^{\infty} \frac{||\Pi_i^n|| |Q_i|}{||\Pi_i+1 X_i|| |\Pi_i X_0||} 
\]

and the right-hand side tends to zero as $n \to \infty$ by the proof of part (i), in particular Eq. (3.27).

Since $Y_i$ is a unit vector with nonnegative entries, $(Y_i, X_i) \geq d^{-1} \min_j X_i^{(j)}$. Hence we also have

\[
\left| \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{\langle e_j, \widetilde{R}_n^i \rangle \langle Y_i, Y_i \rangle |Q_i|}{|\Pi_n X_0|} \right| \leq \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{d}{\min_j X_i^{(j)} |\Pi_i X_0||} |Q_i| 
\]

Thus, to establish (3.30) and consequently part (ii) of the lemma, it is enough to show that

\[
\lim_{n \to \infty} \left| \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle e_j, \Pi_i^n Q_i \rangle}{|\Pi_n X_0|} - \langle e_j, \widetilde{R}_n^i \rangle \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle Y_i, \tilde{Q}_i \rangle |Q_i|}{|\Pi_i X_0||} \right| = 0 \text{ } \hat{P}^\alpha \text{-a.s.} \tag{3.31} \]

Then by the triangle inequality, it is sufficient to show the following.

\textbf{Sublemma 3.7.}

\[
\lim_{n \to \infty} \left| \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle e_j, \Pi_i^n Q_i \rangle}{|\Pi_n X_0|} - \langle e_j, \widetilde{R}_n^i \rangle \frac{\langle L_i^n X_i \rangle}{|\Pi_i X_0||} \right| = 0 \text{ } \hat{P}^\alpha \text{-a.s.} \tag{3.32} 
\]

\[
\lim_{n \to \infty} \left| \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle e_j, \widetilde{R}_n^i \rangle |Q_i|}{|\Pi_n X_0||} \left( \frac{\langle L_i^n + 1, Q_i \rangle}{\langle L_i^n X_i \rangle} - \frac{\langle Y_i + 1, Q_i \rangle}{\langle Y_i + 1, X_i \rangle} \right) \right| = 0 \text{ } \hat{P}^\alpha \text{-a.s.} \tag{3.33} 
\]

and

\[
\lim_{n \to \infty} \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle Y_i + 1, Q_i \rangle}{|\Pi_i X_0||} \left( \langle e_j, \widetilde{R}_n^i \rangle - \langle e_j, \widetilde{R}_n^i \rangle \right) = 0 \text{ } \hat{P}^\alpha \text{-a.s.} \tag{3.34} 
\]

\textbf{Proof of the Sublemma.} First we establish (3.32). To this end, observe by Corollary 1 of Hennion (1997) that

\[
\lim_{n \to \infty} \left( \frac{\Pi_i^n + 1}{||\Pi_i + 1||} - \frac{R_i^n + 1 \otimes L_i^n + 1}{||R_i^n + 1 \otimes L_i^n + 1||} \right) = 0 \text{ } \hat{P}^\alpha \text{-a.s.,} \tag{3.35} 
\]

where $a \otimes b$ denotes the rank-one matrix with the property that

\[
\langle e_i, (a \otimes b) e_j \rangle = \langle e_i, a \rangle \langle b, e_j \rangle. \tag{3.36} 
\]

From (3.35) we infer that

\[
\lim_{n \to \infty} \left( \frac{\langle e_j, \Pi_i^n Q_i \rangle}{||\Pi_i + 1||} - \frac{\langle e_j, \Pi_i^n + 1, Q_i \rangle}{||R_i^n + 1 \otimes L_i^n + 1||} \right) = 0 \tag{3.37} 
\]

and

\[
\lim_{n \to \infty} \left( \frac{\langle \Pi_i^n + 1, Q_i \rangle}{||\Pi_i + 1||} - \frac{\langle \Pi_i^n + 1, Q_i \rangle}{||R_i^n + 1 \otimes L_i^n + 1||} \right) = 0. \tag{3.38} 
\]
Combining (3.37) and (3.38), we conclude that
\[
\lim_{n \to \infty} \frac{\langle e_j, \Pi_{n+1} Q_i \rangle}{\|\Pi_n X_0\|} = \lim_{n \to \infty} \frac{\langle e_j, \Pi_{n+1} Q_i \rangle}{\|\Pi_{n+1} X_i\|} \Pi_i X_0
= \lim_{n \to \infty} \langle e_j, \tilde{R}_n \rangle \frac{\langle L_{n+1}^i, Q_i \rangle}{\langle L_{n+1}^i, X_i \rangle} \Pi_i X_0 \quad \text{\(\widehat{\mathbb{P}}\)-a.s.,} \tag{3.39}
\]
showing, in particular, that the individual terms in (3.32) converge to zero \(\widehat{\mathbb{P}}\)-a.s.

To prove that the sum in (3.32) converges to zero, we now invoke a dominated convergence argument. Since \(\mathcal{R}\) is finite a.s., it suffices to focus on summands with \(i \geq \mathcal{R}\), where we can assume that all components of \(X_i\) are positive, as the remaining terms form a finite sum. Observe that
\[
\frac{\langle L_{n+1}^i, 1 \rangle \max_j Q_i^{(j)} / \langle L_{n+1}^i, 1 \rangle \min_j X_i^{(j)}}{\langle L_{n+1}^i, 1 \rangle \min_j X_i^{(j)}} \leq \frac{|Q_i|}{\min_j X_i^{(j)}},
\]
and therefore
\[
\sup_n \sum_{i=\mathcal{R}}^{n/2} \left| \frac{\langle e_j, \Pi_n X_0 \rangle}{\|\Pi_n X_0\|} - \langle e_j, \tilde{R}_n \rangle \frac{\langle L_{n+1}^i, Q_i \rangle}{\langle L_{n+1}^i, X_i \rangle} \right|
\leq 2 \sup_n \sum_{i=\mathcal{R}}^{n/2} \frac{1}{\min_j X_i^{(j)} \|\Pi_n X_0\|} < \infty \quad \text{\(\widehat{\mathbb{P}}\)-a.s.} \tag{3.41}
\]
by part (i) (where we have used the calculation in (3.25) to handle the first term on the left-hand side). Thus, using a dominated convergence argument to interchange summation and limit (applied pointwise on the space where (3.38) and (3.41) hold), we now conclude that (3.32) follows from (3.38).

Next we turn to (3.33). It follows by Lemma 3.3 of Hennion (1997) that, under \(\widehat{\mathbb{P}}\), the sequence \(\{L_{n+1}^i\}\) converges a.s. as \(n \to \infty\) to \(Y_{i+1}\). Hence, by a dominated convergence argument (analogous to the one just used to establish (3.32)), we conclude that (3.33) holds.

Finally, to establish (3.34), note by Proposition 3.1 of Hennion (1997) that
\[
\left| \tilde{R}_n - R_n \right| = \left| (\Pi_{i+1} R_{n+1}) - (\Pi_n R_n) \right| \leq 2c(\Pi_{i+1}), \tag{3.42}
\]
where \(c(\cdot)\) is bounded above by one and tends to zero \(\widehat{\mathbb{P}}\)-a.s. as \(n-i\) tends to infinity (Hennion (1997), Lemma 3.2). Then (3.34) follows, once again, by a dominated convergence argument. This completes the proof of the sublemma and, consequently, part (ii) of Lemma 3.6.

\(\square\)

**Proof of Lemma 3.6 (continued).** We now return to the proof of main lemma, where it remains to verify that (iii) and (iv) hold.

(iii) Let \(m \in \mathbb{N}\), and set \(B_1 = \max_{x,y} (r_\alpha(x)/r_\alpha(y)) \in (0, \infty)\). Then for \(\alpha > 0\),
\[
\mathbb{E}_{\delta_v}^\alpha \left[ \sup_{n \leq m} |Z_n| 1_{\{r \geq n\}} \right] \leq \mathbb{E}_{\delta_v}^\alpha \left[ \left( \sup_{n \leq m} \left( |v| + \sum_{k=1}^{n} \frac{|\Pi_n^k Q_k|}{\|\Pi_n X_0\|} 1_{\{r \geq k-1\}} \right) \right)^\alpha \right]
\leq (r_\alpha(v))^{-1} \mathbb{E}_{\delta_v}^\alpha \left[ \left( \sup_{n \leq m} \left( |v| + \sum_{k=1}^{n} \frac{|\Pi_n^k Q_k|}{\|\Pi_n X_0\|} 1_{\{r \geq k-1\}} \right) \right)^\alpha \right]
\leq B_1 \mathbb{E}_{\delta_v} \left[ \left( \sup_{n \leq m} \left( |\Pi_n X_0| |v| + \sum_{k=1}^{n} |\Pi_n^k X_n| \cdot |\Pi_n^k Q_k| 1_{\{r \geq k-1\}} \right) \right)^\alpha \right]
\leq B_1 m \mathbb{E}_{\delta_v} \left[ \left( \sum_{k=0}^{m} |\Pi_n^{k+1}| \cdot |\Pi_n^{k+1} Q_k| 1_{\{r \geq k-1\}} \right)^\alpha \right], \text{ where } Q_0 := v,
\]
\[
= B_1 m \mathbb{E}_{\delta_v} \left[ \left( \sum_{k=0}^{m} |\Pi_n^{k+1}| \cdot |\Pi_n^{k+1} Q_k| 1_{\{r \geq k-1\}} \right)^\alpha \right].
\]
Now suppose that $\alpha \geq 1$. Then by Minkowski’s inequality,
\[
\left( \mathbb{E}_v \left[ \left( \sum_{k=0}^{m} \| \Pi_{n+1}^m \| \cdot \| \Pi_{k+1}^n \| \cdot |Q_k|^\alpha 1_{\{\tau \geq k-1\}} \right)^\alpha \right] \right)^{1/\alpha} \\
\leq \sum_{k=0}^{m} \left( \mathbb{E}_v \left[ \| \Pi_{n+1}^m \|^\alpha \cdot \| \Pi_{k+1}^n \|^\alpha \cdot |Q_k|^\alpha 1_{\{\tau \geq k-1\}} \right] \right)^{1/\alpha} \\
= \sum_{k=0}^{m} \left( \mathbb{E} \left[ \| \Pi_{n+1}^m \|^\alpha \right] \right)^{1/\alpha} \left( \mathbb{E} \left[ \| \Pi_{k+1}^n \|^\alpha \right] \right)^{1/\alpha} \left( |v| + \mathbb{E} \left[ |Q_1|^\alpha \right] \right)^{1/\alpha} \mathbb{P}_v (\tau' \geq k-1)^{1/\alpha}.
\]

Now by Corollary 4.6 of Buraczewski et al. (2014), $\mathbb{E} \left[ \| \Pi_n \|^\alpha \right] \leq B_2 \in (0, \infty)$, for all $n$. Moreover, $\mathbb{E} \left[ |Q_i|^\alpha \right] < \infty$ by Hypothesis $(H_2)$, and by our assumption (3.22), $\sup_{v \in F} \mathbb{P} (\tau' \geq k \mid V_0 = v) \leq B_3 t^k$ for some $t \in (0,1)$. Hence by monotone convergence,
\[
\mathbb{E}_\delta \left[ \sup_{n \in \mathbb{N}} |Z_n|^\alpha 1_{\{\tau' \geq n\}} \right] = \lim_{m \to \infty} \mathbb{E}_\delta \left[ \sup_{n \leq m} |Z_n|^\alpha 1_{\{\tau' \geq n\}} \right] \\
\leq B_1 B_2^2 \left( |v| + \mathbb{E} \left[ |Q_1|^\alpha \right] \right) \left( \sum_{k=0}^{\infty} (B_3 t^k)^{1/\alpha} \right)^\alpha < \infty, \quad (3.43)
\]
and this bound is uniform over $v \in F$ for any bounded set $F \subset \mathbb{R}_+^d \setminus \{0\}$.

If $\alpha \leq 1$, then we use the subadditivity, namely the inequality $|x+y|^\alpha \leq |x|^\alpha + |y|^\alpha$ in place of Minkowski’s inequality, and then proceed as before, obtaining an analogous estimate to (3.43), showing again that the left-hand side of (3.43) is finite.

Now it follows from part (ii) that $|Z_n|^\alpha 1_{\{\tau' \geq n\}} \to |Z|^\alpha 1_{\{\tau' = \infty\}}$ $\mathbb{P}^\alpha$-a.s. as $n \to \infty$. Consequently,
\[
\sup_{v \in F} \mathbb{E}_\delta \left[ |Z|^\alpha 1_{\{\tau' = \infty\}} \right] = \sup_{v \in F} \mathbb{E}_\delta \left[ \lim_{n \to \infty} |Z_n|^\alpha 1_{\{\tau' \geq n\}} \right] \leq \sup_{v \in F} \mathbb{E}_\delta \left[ \sup_{n \in \mathbb{N}} |Z_n|^\alpha 1_{\{\tau' \geq n\}} \right] < \infty.
\]

(iv) The almost sure convergence $|Z_n|^\alpha 1_{\{\tau' \geq n\}} \to |Z|^\alpha 1_{\{\tau' = \infty\}}$ was obtained in part (ii), and it was shown in part (iii) that the sequence $\{ |Z_n|^\alpha 1_{\{\tau' \geq n\}} \}_{n \in \mathbb{N}}$ is uniformly integrable, and the $L^1$-convergence follows.

\[\square\]

### 3.3 Markov nonlinear renewal theory

We conclude this section by applying Markov nonlinear renewal theory, as developed by Melfi (1992, 1994), to the processes $\{V_n\}$ and $\{V_n^A\}$, where
\[
V_n^A := \frac{V_n}{d_A(V_n)}, \quad n \in \mathbb{N}, \quad (3.44)
\]
and where we assume here that $d_A$ is bounded and continuous. Melfi’s theory allows us to compare the overjump distributions for these two processes to those of $\{(X_n, S_n)\}$ and $\{(X_n, S_n^A)\}$, respectively, where $S_n^A = S_n - \log d_A(X_n)$.

We begin by verifying the conditions of Kesten’s renewal theorem for the Markov random walks $\{(X_n, S_n)\}$ and $\{(X_n, S_n^A)\}$ under the $\alpha$-shifted measure. We start by quoting these conditions as they are stated in Kesten (1974), with notation adapted for the process $\{(X_n, S_n)\}$.

**I.1** There exists a measure $\eta_\alpha$ on $\mathbb{S}_{d-1}^+$ which is invariant for $\{X_n\}$, and $\mathbb{P}_x^\alpha(X_n \in E, \text{ for some } n) = 1$ for all $x \in \mathbb{S}_{d-1}^+$ and all open sets $E \subset \mathbb{S}_{d-1}^+$ with $\eta_\alpha(E) > 0$.

**I.2** $\mathbb{E}_\eta_\alpha [S_1 - S_0]$ exists and $\lim_{n \to \infty} n^{-1} S_n = \mathbb{E}_\eta_\alpha [S_1 - S_0] > 0 \mathbb{P}^\alpha$-a.s.

**I.3** There exists a sequence $\{\zeta_i\} \subset \mathbb{R}$ such that the group generated by $\{\zeta_i\}$ is dense in $\mathbb{R}$, and such that for each $\zeta_i$ and each $\delta > 0$, there exists $y = y(i, \delta) \in \mathbb{S}_{d-1}^+$ with the following property: For
each $\varepsilon > 0$, there exists $E \subset S_{+}^{d-1}$ with $\eta_{\alpha}(E) > 0$, $m_{1}, m_{2} \in \mathbb{N}_{+}$, and $\gamma \in \mathbb{R}$ such that for any $x \in E$,

$$
\mathbb{P}_{x}^{\eta_{\alpha}}(|X_{m_{1}} - y| < \varepsilon, |S_{m_{1}} - \gamma| \leq \delta) > 0 \quad \text{and} \quad (3.45)
$$

$$
\mathbb{P}_{x}^{\eta_{\alpha}}(|X_{m_{2}} - y| < \varepsilon, |S_{m_{2}} - \gamma - \zeta| \leq \delta) > 0. \quad (3.46)
$$

I.4 For each fixed $x_{0} \in S_{+}^{d-1}$ and $\varepsilon > 0$, there exists $r = r(x_{0}, \varepsilon)$ such that for all bounded measurable functions $f : (S_{+}^{d-1} \times \mathbb{R})^{N} \to \mathbb{R}$ and for all $y \in B_{r}(x_{0})$,

$$
\mathbb{E}_{y}^{\alpha}[f(X, 0, S_{1}, S_{2}, \ldots)] \leq \mathbb{E}_{y}^{\alpha}[f^{x}(X, 0, S_{1}, S_{2}, \ldots)] + \varepsilon|f|_{\infty} \quad \text{and} \quad (3.47)
$$

$$
\mathbb{E}_{y}^{\alpha}[f(X, 0, S_{1}, S_{2}, \ldots)] \leq \mathbb{E}_{y}^{\alpha}[f^{x}(X, 0, S_{1}, S_{2}, \ldots)] + \varepsilon|f|_{\infty}. \quad (3.48)
$$

where $f^{x}(x_{0}, s_{0}, x_{1}, s_{1}, \ldots) := \sup \{f(y_{0}, t_{0}, y_{1}, t_{1}, \ldots) : |x_{i} - y_{i}| + |s_{i} - t_{i}| < \varepsilon \quad \text{for all} \quad i \in \mathbb{N}\}$.

Let us give a brief interpretation of these conditions. Condition I.1 is weaker than $\eta_{\alpha}$-irreducibility, for only $\eta_{\alpha}$-positive open sets are required to be reachable from any initial state. Condition I.2 is the classical assumption of positive drift. Condition I.3 is the implementation of the non-arithmeticity condition; while Condition I.4 states that if $x \to y$, then $\mathbb{P}_{y}^{\alpha}((X_{n}, S_{n})_{n \in \mathbb{N}} \in \cdot)$ converges to $\mathbb{P}_{y}^{\alpha}((X_{n}, S_{n})_{n \in \mathbb{N}} \in \cdot)$ in Prokhorov distance; for details, see Meili (1994), Section 2.2.

**Lemma 3.8.** Assume (H1) and (H2). Then Conditions I.1 – I.4 are satisfied by $\{(X_{n}, S_{n}) : n \in \mathbb{N}\}$. If $d_{A}$ is bounded and continuous on $S_{+}^{d-1}$, then these conditions are also satisfied by $\{(X_{n}, S_{n}^{A}) : n \in \mathbb{N}\}$.

**Proof.** Hypotheses (H1) and (H2) imply the assumptions of Proposition 1 in Kesten (1973), where the validity of I.1 – I.4 is proved for the Markov random walk $\{(X_{n}, S_{n})\}$.

We now verify I.1 – I.4 for the process $\{(X_{n}, S_{n}^{A})\}$. I.1 remains valid, as it only concerns $\{X_{n}\}$. Next observe that $S_{n}^{A} = S_{n} - \log d_{A}(X_{n})$ is a measurable function of $(X_{n}, S_{n})$. Hence any measurable function $f(X_{0}, S_{0}^{A}, X_{1}, S_{1}^{A}, \ldots)$ can be transformed into a measurable function of $\{(X_{n}, S_{n})\}$, and thus I.4 holds as well [using that $d_{A}$ is bounded from above and below]. Since I.3 holds for $\{(X_{n}, S_{n})\}$, it holds for $\{(X_{n}, S_{n}^{A})\}$ as well by replacing $\gamma$ with $\gamma' := \gamma - \log d_{A}(y)$. Finally, I.2 follows by applying Lemma 2.4, which gives

$$
\lim_{n \to \infty} \frac{S_{n}^{A}}{n} = \frac{1}{n} \sum_{i=1}^{n} (S_{i} - S_{i-1}) - (\log d_{A}(X_{1}) - \log d_{A}(X_{i-1})) + \frac{1}{n}\log d_{A}(X_{0})
$$

$$
= \mathbb{E}_{\eta_{\alpha}}^{\eta_{\alpha}}[S_{1} - S_{0} - (\log d_{A}(X_{1}) - \log d_{A}(X_{0}))] = \mathbb{E}_{\eta_{\alpha}}^{\eta_{\alpha}}[S_{1}^{A} - S_{0}^{A}] \quad \mathbb{P}^{\alpha}-\text{a.s.} \quad \Box
$$

Write $\Gamma_{u}^{A} = \inf\{n \in \mathbb{N}_{+} : S_{n}^{A} > \log u\}$ and $\Gamma_{u} = \inf\{n \in \mathbb{N}_{+} : S_{n} > \log u\}$. Then Kesten’s renewal theorem (Kesten (1974), Theorem 1) yields the joint asymptotics of the overjump above level $\log u$ and the position of $X_{n}$ at the time of the overjump.

**Theorem 3.9** (Kesten, 1974). Assume (H1) and (H2). Then:

(i) There is a probability measure $\varrho$ on $S_{+}^{d-1} \times (0, \infty)$, such that, for all $x \in S_{+}^{d-1}$ and all functions $f \in \mathcal{C}_{b}(S_{+}^{d-1} \times (0, \infty))$,

$$
\lim_{u \to \infty} \mathbb{E}_{x}^{\alpha}[f(X_{\Gamma_{u}^{A}}, S_{\Gamma_{u}^{A}} - \log u)] = \int_{S_{+}^{d-1} \times \mathbb{R}_{+}} f(y, s) \varrho(dy, ds).
$$

(ii) If the function $d_{A}$ is bounded and continuous on $S_{+}^{d-1}$, then there is a probability measure $\varrho^{A}$ on $S_{+}^{d-1} \times (0, \infty)$, such that for all $x \in S_{+}^{d-1}$ and all $f \in \mathcal{C}_{b}(S_{+}^{d-1} \times (0, \infty))$,

$$
\lim_{u \to \infty} \mathbb{E}_{x}^{\alpha}[f(X_{\Gamma_{u}^{A}}, S_{\Gamma_{u}^{A}}^{A} - \log u)] = \int_{S_{+}^{d-1} \times \mathbb{R}_{+}} f(y, s) \varrho^{A}(dy, ds).
$$
A representation of \( q \) in terms of the ascending ladder height process is given in Eq. (5.3) below.

The essential result we will need is the nonlinear Markov renewal theorem by Melfi (1992, 1994), which extends Kesten’s renewal theorem to a wider class of processes which are asymptotically “close” to the Markov random walk \( \{(X_n, S_n) : n \in \mathbb{N}\} \).

Let \( \{(Y_n, W_n) : n \in \mathbb{N}\} \) be a stochastic process on \( \mathbb{S}^{d-1}_+ \times \mathbb{R} \), adapted to a filtration \( \mathcal{F}_n \), and throughout the remainder of this section (with a slight abuse of notation) set

\[
T_u := \inf\{n \in \mathbb{N} : W_n > \log u\},
\]

\[
\mathfrak{T}_u := \inf\{n \in \mathbb{N} : S_n > \log u\},
\]

\[
(Y_{u,k}, W_{u,k}) := (Y_{T_u+k}, W_{T_u+k} - W_{T_u}) \quad \text{for } k \geq 0.
\]

Then we have the following (Melfi (1994), Theorem 3).

**Theorem 3.10** (Melfi, 1994). Let \( \{(X_n, S_n) : n \in \mathbb{N}\} \) be a Markov random walk satisfying the assumptions of Kesten’s renewal theorem under \( \mathbb{P}^\alpha \). Assume the following conditions hold:

- (I) For all \( m \geq 1 \), the Prokhorov distance

\[
\delta \left( \mathbb{P}^\alpha \left( (Y_{u,k}, W_{u,k})_{1 \leq k \leq m} \in \cdot \mid \mathcal{F}_{T_u} \right), \mathbb{P}^\alpha_\mathcal{F}_{T_u} \left( (X_k, S_k)_{1 \leq k \leq m} \in \cdot \right) \right)
\]

converges to 0 in \( \mathbb{P}^\alpha \)-probability.

- (II) \( \{W_{T_u} - \log u\}_{u \geq 1} \) is tight under \( \mathbb{P}^\alpha \).

- (III) \( \{Y_{T_u}\}_{u \geq 1} \) is tight under \( \mathbb{P}^\alpha \).

Let \( \varrho \) denote the asymptotic overjump distribution of \( \{(X_n, S_n) : n \in \mathbb{N}\} \) obtained in Theorem 3.9.

Then for all \( f \in \mathcal{C}_b \left( \mathbb{S}^{d-1}_+ \times (0, \infty) \right) \) and all \( x \in \mathbb{S}^{d-1}_+ \),

\[
\lim_{u \to \infty} \mathbb{E}^\alpha_x \left[ f(Y_{T_u}, W_{T_u} - \log u) \right] = \int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}_+} f(y, s) \varrho(dy, ds).
\]

Next, we verify the assumptions of Melfi’s theorem for the process \( \{(Y_n, W_n)\} \) chosen specifically as \( \{(V_n, \log |V_n|)\} \) and \( \{(V_n^A, \log |V_n^A|)\} \), respectively. Note that in this case \( \mathcal{F}_n = \mathcal{F}_n \), and condition (III) is always satisfied since \( Y_{T_u} \in \mathbb{S}^{d-1}_+ \), which is compact.

**Lemma 3.11.** Assume \((H_1)\) and \((H_2)\). Then Condition (I) is satisfied by the pair \( \{(\tilde{V}_n, \log |V_n|)\}, \{(X_n, S_n)\} \).

**Proof.** Using the Markov property,

\[
\mathbb{P}^\alpha \left( (Y_{u,k}, W_{u,k})_{1 \leq k \leq m} \in \cdot \mid \mathcal{F}_{T_u} \right) = \mathbb{P}^\alpha \left( \left( \tilde{V}_{T_u+k}, \log |V_{T_u+k} - \log |V_{T_u}| \right)_{1 \leq k \leq m} \in \cdot \mid X_{T_u}, V_{T_u} \right) = \mathbb{P}^\alpha_{X_{T_u}, V_{T_u}} \left( \left( \tilde{V}_k, \log |V_k - \log |V_0| \right)_{1 \leq k \leq m} \in \cdot \right).
\]

By Guivarc’h and Le Page (2016), Lemma 3.5, the total variation distance between \( \mathbb{P}^\alpha_{x,v} \) and \( \mathbb{P}^\alpha_{y,v} \) is bounded above by \( B|x - y|^\bar{\alpha} \) for some finite constant \( B \), where \( \bar{\alpha} = \min\{\alpha, 1\} \). [Their proof is in the setting of invertible matrices, but carries over to the present setting of nonnegative matrices without change.] Convergence in total variation implies convergence in the Prokhorov metric \( \delta \), and thus

\[
\delta \left( \mathbb{P}^\alpha_{X_{T_u}, V_{T_u}} \left( \left( \tilde{V}_k, \log |V_k - \log |V_0| \right)_{1 \leq k \leq m} \in \cdot \right), \mathbb{P}^\alpha_{V_{T_u}, V_{T_u}} \left( (X_k, S_k)_{1 \leq k \leq m} \in \cdot \right) \right) \leq B \left| X_{T_u} - \tilde{V}_{T_u} \right|^\bar{\alpha} + \delta \left( \mathbb{P}^\alpha_{V_{T_u}, V_{T_u}} \left( \left( \tilde{V}_k, \log |V_k - \log |V_0| \right)_{1 \leq k \leq m} \in \cdot \right), \mathbb{P}^\alpha_{V_{T_u}} \left( (X_k, S_k)_{1 \leq k \leq m} \in \cdot \right) \right).
\]
We begin by considering the second term. Fix the initial values \((\tilde{V}_{T_u}, V_{T_u}) = (\tilde{v}, v)\), and introduce the notation \(V_k^{(0)} := \sum_{j=1}^{k} M_k \cdots M_{j+1} Q_j\), \(k = 2, 3, \ldots\) and \(V_1^{(0)} = Q_1\). Then for any \(k \in \mathbb{N}_+\),

\[
|\log |V_k| - \log |V_0| - S_k| = \log \left| \frac{\Pi_k v + V_k^{(0)}}{\Pi_k v v} \right| \leq \log \left( 1 + \frac{|V_k^{(0)}|}{|\Pi_k v|} \right) \leq \frac{|V_k^{(0)}|}{|\Pi_k v|} \tag{3.52}
\]

and

\[
|\tilde{V}_k - X_k| = \left| \frac{V_k}{|\tilde{V}_k|} - \frac{V_k}{|\Pi_k v|} + \frac{V_k}{|\Pi_k v|} - \frac{\Pi_k v}{|\Pi_k v|} \right| \leq \left| \frac{\Pi_k v}{|\Pi_k v|} - |V_k| \right| \geq \frac{|V_k^{(0)}|}{|\Pi_k v|} \leq \frac{2|V_k^{(0)}|}{|\Pi_k v|}, \tag{3.53}
\]

using the triangle inequality. Thus for all \(v \in \mathbb{R}^d_+ \setminus \{0\}\),

\[
P_\alpha \left( \left( |\tilde{V}_k, \log |V_k| - \log |V_0||_{1 \leq k \leq m} - (X_k, S_k)_{1 \leq k \leq m} \right|_{\infty} \geq \varepsilon \bigg| V_0 = v \right) \leq \frac{2m}{\varepsilon} \sum_{k=1}^{m} \mathbb{P}_\alpha \left( \frac{|V_k^{(0)}|}{|\Pi_k v|} \geq \varepsilon \right) \leq \left( \frac{2m}{\varepsilon} \right)^\alpha \sum_{k=1}^{m} \mathbb{E} \left[ \frac{|V_k^{(0)}|^\alpha}{|\Pi_k v|^{\alpha}} \right], \tag{3.54}
\]

for some universal constant \(B\), where we used Chebyshev’s inequality, the definition of the \(\alpha\)-shifted measure, and the boundedness of \(r_\alpha\) in the last identity. Hence

\[
\lim_{u \to \infty} \sup_{v : |v| \geq u} \mathbb{P}_\alpha \left( \left( \tilde{V}_k, \log |V_k| - \log |V_0| |_{1 \leq k \leq m} - (X_k, S_k)_{1 \leq k \leq m} \right|_{\infty} \geq \varepsilon \bigg| V_0 = v \right) = 0.
\]

Recall that convergence in probability implies weak convergence, which is equivalent to convergence in the Prokhorov metric. Since \(\mathbb{P}_\alpha(T_u < \infty) = 1\) and \(|V_{T_u}| \geq u\), we conclude that

\[
\lim_{u \to \infty} \mathcal{D} \left( \mathbb{P}_\alpha \bigg|_{V_{T_u}, V_{T_u}} \left( \tilde{V}_k, \log |V_k| - \log |V_0| |_{1 \leq k \leq m} \right|_{\infty} \geq \varepsilon \bigg| (X_k, S_k)_{1 \leq k \leq m} \right), \mathbb{P}_\alpha \left( (X_k, S_k)_{1 \leq k \leq m} \right) \right) = 0 \quad \text{\(P_\alpha\)-a.s.}
\]

To complete the proof, we observe that the first term on the right-hand side of (3.51) tends to zero in \(P_\alpha\)-probability by Lemma 3.12, which is given next.

\[ \tag{Lemma 3.12} \\]
for a function \( c(\cdot) \) is bounded above by one and tends to zero \( \hat{P}^\alpha \)-a.s. as \( n-i \) tends to infinity (Hennion (1997), Lemma 3.2). Since \( \mathbb{P}_x^\alpha \) is absolutely continuous with respect to the measure \( \hat{P}^\alpha \) (Buraczewski et al. (2014), Lemma 6.2), it follows that \( c\left(\Pi_{T_{u+1}}^x\right) \to 0 \mathbb{P}_x^\alpha \)-a.s. for all \( x \in S_{d-1}^d \), and hence

\[
I_2(u) \leq \mathbb{P}_x^\alpha \left( 2c\left(\Pi_{T_{u+1}}^x\right) > \frac{\varepsilon}{2} \right) \to 0 \text{ as } u \to \infty. \quad (3.56)
\]

Now consider \( I_1(u) \) as \( u \to \infty \). Repeating the calculation in (3.53) yields

\[
I_1(u) \leq E_x^\alpha_{V_{\tau_w}} \left( \frac{4|V_n(0)|}{u|\Pi_{T_u}V_0|} > \frac{\varepsilon}{2} \right). \quad (3.57)
\]

Now recall from Lemma 3.6 (i) that

\[
F^0 := \sup_{n \in \mathbb{N}} \frac{|V_n(0)|}{|M_n \cdots M_1 X_0|} < \infty \quad \hat{P}^\alpha \text{-a.s.} \quad (3.58)
\]

Using that \( \mathbb{P}_x^\alpha_{V_{\tau_w}} \leq B \hat{P}^\alpha \) for some universal constant \( B \) by Lemma 6.2 of Buraczewski et al. (2014), we obtain

\[
I_1(u) \leq \lim_{u \to \infty} \hat{P}^\alpha \left( F^0 > \frac{\varepsilon}{8} \right) = 0. \quad (3.59)
\]

It remains to check condition (II) of Melfi’s theorem.

**Lemma 3.13.** \( \{W_{T_u} - \log u\}_{u \geq 1} \) is tight under \( \mathbb{P}^\alpha \).

**Proof.** A sufficient condition is given in Melfi (1994), Section 5.2. Letting \( \xi_n := \log |V_n| - S_n \) and supposing that \( \{\xi_{T_u}\}_{u \geq 1} \) and \( \{\xi_{S_u}\}_{u \geq 1} \) are tight under \( \mathbb{P}^\alpha \), then it follows that \( \{W_{T_u} - \log u\}_{u \geq 1} \) is tight. Now by Lemma 3.6,

\[
\xi_n = \log \frac{|V_n|}{|\Pi_{T_u}V_0|} \to \log Z \quad \mathbb{P}^\alpha \text{-a.s.,}
\]

for a finite random variable \( Z \). Since \( T_u \) and \( S_u \) are stopping times with respect to the filtration \( \{\mathcal{F}_n\} \) and tend to infinity as \( u \to \infty \), we deduce that

\[
\lim_{u \to \infty} \xi_{T_u} = \log Z \quad \mathbb{P}^\alpha \text{-a.s. and} \quad \lim_{u \to \infty} \xi_{S_u} = \log Z \quad \mathbb{P}^\alpha \text{-a.s.}
\]

Thus, in particular, the families \( \{\xi_{T_u}\}_{u \geq 1} \) and \( \{\xi_{S_u}\}_{u \geq 1} \) converge in distribution under \( \mathbb{P}^\alpha \) and are consequently tight. \( \square \)

We are now in a position to apply Melfi’s theorem.

**Theorem 3.14.** Assume \((H_1)\) and \((H_2)\). Then for all \( f \in \mathcal{C}_b\left(S_{d-1}^d \times (0, \infty)\right) \) and all \( x \in S_{d-1}^d \) and \( u \in \mathbb{R}_+ \setminus \{0\} \),

\[
\lim_{u \to \infty} E_x^\alpha \left[ f \left( \tilde{V}_{T_u}, \log \frac{|V_{T_u}|}{u} \right) \right] = \int_{S_{d-1}^d \times \mathbb{R}_+} f(y, s) \phi(dy, ds). \quad (3.60)
\]

Further, if \( d_A \) is bounded and continuous on \( S_{d-1}^d \), then for all \( f \in \mathcal{C}_b\left(S_{d-1}^d \times (0, \infty)\right) \) and all \( x \in S_{d-1}^d \) and \( v \in \mathbb{R}_+ \setminus \{0\} \),

\[
\lim_{u \to \infty} \mathbb{P}_x^\alpha \left[ f \left( \tilde{V}_{T_u}^A, \log \frac{|V_{T_u}^A|}{u} \right) \right] = \int_{S_{d-1}^d \times \mathbb{R}_+} f(y, s) \phi_A(dy, ds). \quad (3.61)
\]
Proof. Condition (III) is necessarily satisfied since \( S_{n+1} \) is compact, and for the process \( \{ (\hat{V}_n, \log |\hat{V}_n|) \} \), the validity of Conditions (I) and (II) has been proved in Lemmas 3.11 and 3.13, respectively. Thus (3.60) follows from Theorem 3.10.

Turning to (3.61), we need to check the validity of Conditions (I) and (II) for \( \{ (\hat{V}_n, \log |\hat{V}_n|) \} \). Recall that \( V_n^A := V_n/d_A(\hat{V}_n) \), implying that \( \hat{V}_n = V_n \) (since these two quantities have the same direction). Moreover \( S_n^A = S_n - \log d_A(X_n) \). Thus, for \( f(x, s) = (x, s - \log d_A(x)) \), we have that \( \{ (V_n^A, \log |V_n^A|) \} = \{ f(\hat{V}_n, \log |\hat{V}_n|) \} \) and \( \{ (X_n, S_n^A) \} = \{ f(X_n, S_n) \} \). Hence, for the process \( \{ (V_n^A, \log |V_n^A|) \} \), Condition (I) can be deduced from Lemma 3.11. Finally, since \( d_A \) is bounded, the tightness of \( \{ \log |V_{n+1}^A| - \log \alpha \} = \log |V_{n+1}^A| - \log d_A(V_{n+1}^A) - \log \alpha \) follows, in the same way, from Lemma 3.13. Thus we conclude (3.61). \( \square \)

We conclude with a result concerning the first passage times in the \( \alpha \)-shifted measure.

Lemma 3.15. Assume that \( d_A \) is bounded and continuous. Then
\[
\lim_{u \to \infty} \frac{T_u^A}{\log u} = \frac{1}{\lambda(\alpha)} \quad \text{in } \mathbb{P}^\alpha\text{-probability.} \tag{3.62}
\]

Proof. By definition, \( V_n = Z_n e^{S_n} \) and \( V_n^A = V_n/d_A(\hat{V}_n) \), and consequently
\[
\log |V_n^A| = S_n + |Z_n| - \log d_A(\hat{V}_n) := S_n + \xi_n. \tag{3.63}
\]

Also, it follows by definition that \( T_u^A := \inf \{ n : V_n \in A \} = \inf \{ n : |V_n| > d_A(\hat{V}_n) u \} = \inf \{ n : \log |V_n^A| > \log u \} \). Now recall that sup\( n \in \mathbb{N} \) \( Z_n \) is finite a.s., by Lemma 3.6. Since \( d_A \) is bounded, it follows that the sequence \( \{ \xi_n \} \) in (3.63) satisfies
\[
\lim_{n \to \infty} \frac{1}{n} \left( \max_{1 \leq k \leq n} \xi_k \right) = 0 \quad \mathbb{P}^\alpha\text{-a.s.,}
\]
i.e. \( \{ \xi_n \} \) is slowly changing (as defined in Siegmund (1985), Eq. (9.5)). Moreover, by Lemma 2.3, \( S_n/n \to \lambda(\alpha) \) a.s., and hence \( \log |V_n^A|/n \to \lambda(\alpha) \) a.s. The result then follows by reasoning as in Siegmund (1985), Lemma 9.13. \( \square \)

4 Characterizing the large exceedances over cycles

Before turning to the proofs of the main theorems of the paper, we first establish a few required results concerning the behavior of the post-\( T_u^A \) process. The central results of this section are Proposition 4.1—which will be used throughout the paper—and Proposition 4.9, which will be the basis for the proof of Theorem 2.5 in the next section.

Recall that \( \tau \) denotes the return time to a set \( D = B_\tau(0) \cap \mathbb{R}^d_+ \setminus \{ 0 \} \) for \( \pi(D) > 0 \), and that
\[
T_u^A := \inf \{ n : V_n \in uA \} = \inf \{ n : |V_n| > u \}, \quad \text{where}
\]
\[
V_n^A := \frac{V_n}{d_A(\hat{V}_n)}, \quad n = 0, 1, \ldots \tag{4.1}
\]

Also recall that
\[
r_n^A(x) = r_\alpha(x) (d_A(x))^\alpha, \quad x \in \mathbb{R}^{d-1}. \tag{4.2}
\]

Finally, we say that a function \( g : (\mathbb{R}^d_+)^{m+1} \to \mathbb{R} \) is almost \( \theta \)-Hölder continuous if
\[
g(v_0, \ldots, v_m) = \hat{g}(v_0, \ldots, v_m) 1_{\{|v_m| \geq \delta\}} \tag{4.3}
\]
for some \( \delta \geq 0 \) and \( \theta \)-Hölder continuous \( \hat{g} \).

Much of this section will be devoted to the proof of the following proposition, which can be viewed as a generalization of Collamore and Vidyashankar (2013b), Proposition 6.1, to the setting of matrix recursions.
Proposition 4.1. Assume Hypotheses \((H_1)\) and \((H_2)\) are satisfied. Let \(m \in \mathbb{N}\) and \(g : \mathbb{R}^d \to \mathbb{R}\) be a bounded almost \(\theta\)-Hölder continuous function for \(\theta \leq \min\{1, \alpha\}\), and assume that the function \(d_A\) is bounded and continuous on \(S^{d-1}_+\). Then for any \(v \in \mathbb{R}^d_+ \setminus \{0\}\),

\[
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ g \left( \frac{V_{T^A(u)}}{u}, \ldots, \frac{V_{T^A(u) + m}}{u} \right) 1_{\{T^A_u < \tau\}} \bigg| V_0 = v \right] = r_\alpha(v) \mathbb{E}_\delta^u \left[ |Z|^\alpha 1_{\{\tau = \infty\}} \right] \int_{S^{d-1}_+ \times \mathbb{R}_+} e^{-\alpha s} r^*_\alpha(x) \mathbb{E}_x \left[ e^{-\alpha s(X_m)} g(e^{S_0} X_0, \ldots, e^{S_m} X_m) \right] g^A(dx, ds). \tag{4.4}
\]

Observe that \(g^A\) is the asymptotic law, as \(u \to \infty\), of \((V_{T^A(u)})^{-1}, \log |V_{T^A(u)}| - \log u\), while on the left-hand side of (4.4), we evaluate the function \(g\) for the process \(V_n\) \((\text{not } \{V^A_n\})\) at a sequence of times commencing at time \(T^A_u\). Taking into account (4.1), this explains the additional summand \(\log d_A(x)\) in the expression for \(S_0\); namely, it arises when transforming \(V^A_{T^A_u}\) into \(V_{T^A_u}\).

An important special case occurs when we take \(m = 0\) and \(g(V_{T^A(u)}/u) \equiv 1\), in which case we obtain:

Corollary 4.2. Assume Hypotheses \((H_1)\) and \((H_2)\) are satisfied and the function \(d_A\) is bounded and continuous on \(S^{d-1}_+\). Then for any \(v \in \mathbb{R}^d_+ \setminus \{0\}\),

\[
\lim_{u \to \infty} u^\alpha \mathbb{P} \left( T^A_u < \tau \bigg| V_0 = v \right) = r_\alpha(v) \mathbb{E}_\delta^u \left[ |Z|^\alpha 1_{\{\tau = \infty\}} \right] \int_{S^{d-1}_+ \times \mathbb{R}_+} e^{-\alpha s} r^*_\alpha(x) \mathbb{E}_x \left[ e^{-\alpha s(X_m)} g(X_0, \ldots, e^{S_m} X_m) \right] g^A(dx, ds). \tag{4.5}
\]

If \(A = \{v \in \mathbb{R}^d_+ : |v| > 1\}\), then \(T^A_u\) reduces to the first exceedance time of \(|V_n|\) above the level \(u\); that is, \(T^A_u = T_u := \inf\{n : |V_n| > u\}\). Furthermore, we then have \(d_A(x) = 1\) and thus \(r_\alpha^A = r_\alpha\).

Consequently, in this case, Proposition 4.1 and Corollary 4.2 hold with \(T^A_u\), \(r_\alpha^A\), \(g^A\) replaced with \(T_u\), \(r_\alpha\), \(g\), respectively. Then we can easily apply the definition of the \(\alpha\)-shifted measure to obtain:

Corollary 4.3. Assume Hypotheses \((H_1)\) and \((H_2)\) are satisfied. Let \(m \in \mathbb{N}\) and \(g : \mathbb{R}^d \to \mathbb{R}\) be a bounded almost \(\theta\)-Hölder continuous function for \(\theta \leq \min\{1, \alpha\}\). Then for any \(v \in \mathbb{R}^d_+ \setminus \{0\}\),

\[
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ g \left( \frac{V_{T^A(u)}}{u}, \ldots, \frac{V_{T^A(u) + m}}{u} \right) 1_{\{T^A_u < \tau\}} \bigg| V_0 = v \right] = r_\alpha(v) \mathbb{E}_\delta^u \left[ |Z|^\alpha 1_{\{\tau = \infty\}} \right] \int_{S^{d-1}_+ \times \mathbb{R}_+} e^{-\alpha s} r^*_\alpha(x) \mathbb{E}_x \left[ e^{-\alpha s(X_m)} g(X_0, \ldots, e^{S_m} X_m) \right] g(dx, ds). \tag{4.6}
\]

To establish Proposition 4.1, we will rely on the following.

Lemma 4.4. Assume the conditions of Proposition 4.1. Then:

(i) For all \(v \in \mathbb{R}^d_+ \setminus \{0\}\), we have the \(L^1\)-convergence

\[
\lim_{n \to \infty} \lim_{u \to \infty} \mathbb{E}_\delta^u \left[ |Z_{T^A_u}|^\alpha 1_{\{T^A_u < \tau\}} - |Z_n|^\alpha 1_{\{n \leq T^A_u\}} 1_{\{n \leq \tau\}} \right] = 0. \tag{4.7}
\]

(ii) Let

\[
\mathcal{G}_u := \frac{1}{r_\alpha(X_{T^A_u})} \left( \frac{V_{T^A_u}}{u} \right) ^{-\alpha} \mathbb{E} \left[ g \left( \frac{V_{T^A_u}}{u}, \ldots, \frac{V_{T^A_u + m}}{u} \right) | \mathcal{F}_{T^A_u} \right], \quad u > 0. \tag{4.8}
\]

Then, independent of \(n\), we have \(\mathbb{P}^a\)-a.s. that

\[
\lim_{u \to \infty} \mathbb{E}^a \left[ \mathcal{G}_u | \mathcal{F}_n \right] 1_{\{n \leq T^A_u\}} = \int_{S^{d-1}_+ \times \mathbb{R}_+} e^{-\alpha s} r^*_\alpha(x) \mathbb{E}_x \left[ e^{-\alpha s(X_m)} g(e^{S_0} X_0, \ldots, e^{S_m} X_m) \bigg| X_0 = x, S_0 = s + \log d_A(x) \right] g^A(dx, ds). \tag{4.9}
\]
Proof of Lemma 4.4. (i) By Lemma 3.2, $\tau$ satisfies the assumptions in Lemma 3.6 (iii). Thus, this result is a direct consequence of Lemma 3.6 (iv), where the $L^1$-convergence $|Z_n|^\alpha \mathbf{1}_{\{n \leq \tau\}} \to |Z|^\alpha \mathbf{1}_{\{\tau = \infty\}}$ is proved. It follows that $|Z_n|^\alpha \mathbf{1}_{\{n \leq \tau\}}$ constitutes a Cauchy sequence in $L^1$, which yields the assertion.

(ii) Let $n \in \mathbb{N}_+$. Then by the Markov property,

$$\mathbb{E}^\alpha \left[ \mathbb{E}_u [ \mathcal{F}_n ] \mathbf{1}_{\{n \leq T^A_n\}} \right] = \mathbb{E}^\alpha_{X_n, V_n} \left[ \mathbb{E}_u \mathbf{1}_{\{n \leq T^A_n\}} \right] \quad \text{P}^\alpha\text{-a.s.}$$

As $\lim_{u \to \infty} \mathbb{1}_{\{n \leq T^A_n\}} = 1$ P$^\alpha$-a.s., it suffices to determine $\lim_{u \to \infty} \mathbb{E}^\alpha_{x,v} [ \mathbb{E}_u ]$ and show that this quantity is independent of $x$ and $v$. For all $v \in \mathbb{R}^d_+$ and $u > 0$, set

$$G_u(v) = \mathbb{E} \left[ g \left( v, \Pi_1 v + \frac{V_0(v)}{u}, \ldots, \Pi_m v + \frac{V_0(v)}{u} \right) \right], \quad G(v) = \mathbb{E} \left[ g \left( v, \Pi_1 v, \ldots, \Pi_m v \right) \right],$$

where $\Pi_k := M_k \cdots M_1$ for $k \geq 1$, and $V^{(0)}_k := \sum_{i=1}^k M_k \cdots M_{i+1} Q_i$ for $k \geq 2$ and $V^{(0)}_1 := Q_1$. Now consider the decomposition:

$$\mathbb{E}^\alpha_{x,v} [ \mathbb{E}_u ] = \mathbb{E}^\alpha_{x,v} \left[ \frac{1}{r_\alpha(X^{T^A}_n)} \left( \left| \frac{V^{T^A}_n}{u} \right| \right)^{-\alpha} G_u \left( \frac{V^{T^A}_n}{u} \right) \right]$$

$$= \mathbb{E}^\alpha_{x,v} \left[ \frac{1}{r_\alpha(X^{T^A}_n)} \left( \left| \frac{V^{T^A}_n}{u} \right| \right)^{-\alpha} \left( G_u \left( \frac{V^{T^A}_n}{u} \right) - G \left( \frac{V^{T^A}_n}{u} \right) \right) \right]$$

$$+ \mathbb{E}^\alpha_{x,v} \left[ \frac{r_\alpha(V^{T^A}_n)}{r_\alpha(X^{T^A}_n)} \frac{1}{r_\alpha(V^{T^A}_n)} \left( \left| \frac{V^{T^A}_n}{u} \right| \right)^{-\alpha} G \left( \frac{V^{T^A}_n}{u} \right) \right] := \mathbb{I}_1(u) + \mathbb{I}_2(u). \quad (4.10)$$

**Step 1.** We begin by showing that $\mathbb{I}_1(u) \to 0$ as $u \to \infty$. Write $g(v_0, \ldots, v_m) = \tilde{g}(v_0, \ldots, v_m)\mathbf{1}_{\{|v_m| \geq \delta\}}$, where $\tilde{g}$ is \(\theta\)-Hölder continuous. Then

$$\left| g \left( \left( \Pi_k v + \frac{V^{(0)}_k}{u} \right)^m \right) \right| \leq \left| \tilde{g} \left( \left( \Pi_k v + \frac{V^{(0)}_k}{u} \right)^m \right) \right| + |\delta| \mathbb{E}^\alpha_{x,v} \left( \left| \Pi_m v + \frac{V^{(0)}_m}{u} \right| \right) - \mathbb{I}_{1,\delta,\infty}(\Pi_m v)$$

for some constant $B_1$ arising from the \(\theta\)-Hölder continuity of $\tilde{g}$. Let $(M^*, Q^*)$ be a pair of random variables which are independent of the sequence $(M_n, Q_n)$, where the law of $(M^*, Q^*)$ is given by $\mathbb{P} ( (M_n, V^{(0)}_m) \in \cdot )$ under $\mathbb{P}^\alpha$. Setting $B_2 := \max_{y \in \mathbb{R}^d_+} (r_\alpha(y))^{-1}$ and using that $(\left| V^{T^A}_n \right|/u)^{-\alpha} < 1$, we obtain that

$$\mathbb{I}_1(u) \leq \frac{1}{u^\theta} B_1 \sum_{k=1}^m \left| V^{(0)}_k \right|^\theta + B_2 |\delta| \mathbb{E}^\alpha_{x,v} \left( \left| M^* \frac{V^{T^A}_n}{u} + Q^* \right| \right) \geq \delta - \mathbb{P}^\alpha_{x,v} \left( \left| M^* \frac{V^{T^A}_n}{u} \right| \geq \delta \right). \quad (4.11)$$

Since the \(\theta\)-moment of $V^{(0)}_k$ is finite, the first term tends to zero as $u \to \infty$. For the second term, we use the $\mathbb{P}^\alpha$-convergence $(M^*, Q^*/u) \Rightarrow (M^*, 0)$ and $(\tilde{V}^{T^A}_n \log |V^{T^A}_n| - \log u) \Rightarrow g^A$ (by Theorem 3.14). Let $(X, S) \sim g^A$ be a random vector independent of $(M^*, Q^*)$ under $\mathbb{P}^\alpha$. Using that $V_n = d_A(\tilde{V}^{T^A}_n) V^{A}_n$ (cf. (3.44)), we have $V^{T^A}_n / u \Rightarrow d_A(X)e^S X$. [Here $X$ describes the limiting direction of $V^{A}_n / u$ and $S$...
the limiting logarithmic overjump, as \( \log |V_{T_a}^A| - \log u \Rightarrow S \). Since the sequences \( \{(M^*, Q^*/u)\} \) and \( \{V_{T_a}^A/u\} \) are independent, they also converge jointly in distribution. Hence, under \( \mathbb{P}^\alpha \),

\[
M^* \frac{V_{T_a}^A}{u} + \frac{Q^*}{u} \Rightarrow d_A(X)e^S M^* X \quad \text{and} \quad M^* \frac{V_{T_a}^A}{u} \Rightarrow d_A(X)e^S M^* X.
\]

Thus, the second term in (4.11) vanishes if \( \delta, \infty \) is a continuity set for \( d_A(X)e^S |M^* X| \).

We now show that \( \delta, \infty \) is a continuity set. Since \( M^* \) is independent of \( (X, S) \), it suffices to show that for any allowable matrix \( m \), \( d_A(X)e^S|mX| = \delta \) has probability 0. Now for each fixed \( y \in \mathbb{S}^{d-1}_+ \), the equation \( h(s) := d_A(y)e^{|ms|} = \delta \) has a unique solution \( s_y \in \mathbb{R} \). Hence

\[
\mathbb{P}^\alpha(d_A(X)e^S|mX| = \delta) = \int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}} \mathbf{1}_{\{s=s_y\}} \varrho^A(dy, ds) = 0,
\]

since the radial component of the overjump distribution is absolutely continuous with respect to Lebesgue measure (as can be seen from the representation of \( \varrho^A \) in Eq. (1.16) of Kesten (1974), or Eq. (5.3) below, which is valid for \( \varrho^A \) upon replacing \( S \) by \( S^A \) everywhere).

Thus, having shown that \( \delta, \infty \) is a continuity set, we conclude by the Portmanteau theorem that for all \( x \in \mathbb{S}^{d-1}_+ \) and \( v \in \mathbb{R}^d \setminus \{0\} \),

\[
\mathbb{P}^\alpha_{x,v}\left(M^* \frac{V_{T_a}^A}{u} + \frac{Q^*}{u} \geq \delta\right) = \mathbb{P}^\alpha_{x,v}\left(M^* \frac{V_{T_a}^A}{u} \geq \delta\right)
\]

and hence also the second member of (4.11) vanishes as \( u \to \infty \). Thus \( \mathbb{I}_1(u) \to 0 \) as \( u \to \infty \).

**Step 2.** Now turn to \( \mathbb{I}_2(u) \). Using Theorem 3.14, again invoke the convergence \( (V_{T_a}^A)^\alpha, \log |V_{T_a}^A| - \log u \Rightarrow \varrho^A \) under \( \mathbb{P}^\alpha \). Moreover, by Lemma 3.12, using the continuity and boundedness of \( r_\alpha \), we have that \( r_\alpha(V_{T_a}^A)/r_\alpha(X_{T_a}) \) tends to one in \( \mathbb{P}^\alpha \)-probability. Hence by Slutsky’s theorem, the quantity inside \( \mathbb{I}_2(u) \) converges in law, and identifying this limit distribution, we deduce that

\[
\lim_{u \to \infty} \mathbb{I}_2(u) = \lim_{u \to \infty} \mathbb{E}_{x,v}^\alpha \left[ \frac{r_\alpha(V_{T_a}^A)}{r_\alpha(X_{T_a})} \frac{1}{r_\alpha(V_{T_a}^A)} \left( \frac{|V_{T_a}^A|}{u} \right)^{-\alpha} G \left( \frac{V_{T_a}^A}{u} \right) \right]
\]

\[
= \int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}^+} \frac{e^{-\alpha s}}{r_\alpha(y)} G \left( d_A(y)e^s y \right) \varrho^A(dy, ds)
\]

\[
= \int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}^+} \frac{e^{-\alpha s}}{r_\alpha(y)} \mathbb{E} \left[ g(e_{S_0}X_0, \ldots, e_{S_m}X_m) \big| X_0 = y, S_0 = s + \log d_A(y) \right] \varrho^A(dy, ds). \quad \square
\]

**The dual change of measure.** Prior to proving Proposition 4.1, we introduce a “dual” change of measure, where the process \( \{(M_n, Q_n) : n = 1, 2, \ldots\} \) follows the \( \alpha \)-shifted measure until the random time \( T_a \), and follows the original measure thereafter. We shall denote expectation relative to this dual measure by \( \mathbb{E}^\beta[\cdot] \). More formally, for any \( n \in \mathbb{N} \), define

\[
\mathbb{E}^\beta[h(V_0, M_1, Q_1, \ldots, M_n, Q_n)|X_0 = x, V_0 = v]
\]

\[
= \mathbb{E} \left[ \frac{M_n \wedge T_a \cdots M_1 x}{r_\alpha(x)} \cdot r_\alpha(X_{T_a}^{M_n}) h(V_0, M_1, Q_1, \ldots, M_n, Q_n) \big| X_0 = x, V_0 = v \right], \quad (4.12)
\]

for all measurable functions \( h : \mathbb{R}^d \times (\mathfrak{M} \times \mathbb{R}^d)^n \to \mathbb{R} \). Following the notational conventions of the previous sections, we write \( \mathbb{E}^\beta_\gamma[\cdot] = \int \mathbb{E}^\beta[\cdot|X_0 = v, V_0 = v] \gamma(du) \) for any probability measure \( \gamma \) on \( \mathbb{R}^d \setminus \{0\} \).
Proof of Proposition 4.1. Note that \( \{V_n\} \) is transient in the \( \alpha \)-shifted measure and thus \( T_{u}^{A} < \infty \) a.s.; cf. Lemma 3.15. Hence, employing the dual change of measure in (4.12) over the random time interval \([0, T_{u}^{A}]\) yields that
\[
u^{a}\mathbb{E}\left[g\left(\frac{V_{T_{u}^{A}}}{u}, \ldots, \frac{V_{T_{u}^{A}+m}}{u}\right)1_{\{T_{u}^{A}<\tau\}}\big| V_{0} = v\right] = \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[g\left(\frac{V_{T_{u}^{A}}}{u}, \ldots, \frac{V_{T_{u}^{A}+m}}{u}\right)1_{\{T_{u}^{A}<\tau\}}\right]. \tag{4.13}\]

Now substitute the quantity \( \mathfrak{G}_{u} \), defined in (4.8), into the previous equation. Noting that
\[
Z_{n} := \frac{V_{n}}{M_{n} \cdots M_{1}X_{0}} := \frac{V_{n}}{e^{S_{n}}} \quad n \in \mathbb{N}_{+},
\]
we obtain after a little algebra that
\[
u^{a}\mathbb{E}\left[g\left(\frac{V_{T_{u}^{A}}}{u}, \ldots, \frac{V_{T_{u}^{A}+m}}{u}\right)1_{\{T_{u}^{A}<\tau\}}\big| X_{0} = \bar{\nu}, V_{0} = v\right] = \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[Z_{T_{u}^{A}}^{\alpha}\mathfrak{G}_{u}1_{\{T_{u}^{A}<\tau\}}\right]. \tag{4.14}\]

The right-hand side can be further equated, for \( n \in \mathbb{N}_{+} \), to
\[
\nu^{a}\mathbb{E}_{\delta_{\alpha}}\left(\left|Z_{T_{u}^{A}}^{\alpha}1_{\{T_{u}^{A}<\tau\}} - |Z_{n}^{\alpha}|1_{\{n \leq T_{u}^{A}\}}1_{\{n \leq \tau\}}\right| \mathfrak{G}_{u}\right) + \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[Z_{n}^{\alpha}1_{\{n \leq T_{u}^{A}\}}1_{\{n \leq \tau\}}\right], \tag{4.15}\]
where we have replaced \( \mathbb{E}_{\delta_{\alpha}}[^{\cdot}\mathfrak{F}_{n}] \) with \( \mathbb{E}^{\alpha}[^{\cdot}\mathfrak{F}_{n}] \) in the last expectation, since this conditional expectation depends only on \((X_{n}, V_{n})\), and not on the initial values \((X_{0}, V_{0}) \) once \((X_{n}, V_{n}) \) has been specified. Moreover, we have replaced the dual change of measure by the \( \alpha \)-shifted measure, since they coincide for random variables which are \( \mathfrak{F}_{T_{u}^{A}} \)-measurable.

To analyze the quantity in (4.14), we first take the limit as \( u \to \infty \) and then as \( n \to \infty \). By part (i) of Lemma 4.4 and the boundedness of \( \mathfrak{G}_{u} \), we deduce from (4.13) and (4.14) that
\[
\lim_{u \to \infty} \nu^{a}\mathbb{E}\left[g\left(\frac{V_{T_{u}^{A}}}{u}, \ldots, \frac{V_{T_{u}^{A}+m}}{u}\right)1_{\{T_{u}^{A}<\tau\}}\big| V_{0} = v\right] = \lim_{n \to \infty} \lim_{u \to \infty} \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[Z_{n}^{\alpha}1_{\{n \leq T_{u}^{A}\}}1_{\{n \leq \tau\}}\right]. \tag{4.16}\]

Now by Lemma 3.6 (iii), \( \{|Z_{n}^{\alpha}|1_{\{n \leq \tau\}}\} \) is uniformly integrable. Denote by \( \mathfrak{S} \) the right-hand side of (4.9). Since \( \mathfrak{G}_{u} \) is bounded by \( b^{-1} |g|_{\infty} \) and \( T_{u}^{A} \uparrow \infty \) \( \mathbb{P}^{\alpha} \)-a.s., it follows by Lemma 4.4 (ii) that
\[
\lim_{u \to \infty} \nu^{a}\mathbb{E}\left[g\left(\frac{V_{T_{u}^{A}}}{u}, \ldots, \frac{V_{T_{u}^{A}+m}}{u}\right)1_{\{T_{u}^{A}<\tau\}}\big| V_{0} = v\right] = \lim_{n \to \infty} \lim_{u \to \infty} \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[Z_{n}^{\alpha}1_{\{n \leq T_{u}^{A}\}}1_{\{n \leq \tau\}}\right] = \nu^{a}\alpha(\bar{\nu})\mathbb{E}_{\delta_{\alpha}}\left[Z^{\alpha}1_{\{\tau = \infty\}}\right] \mathfrak{S}, \tag{4.17}\]
where \( \mathfrak{S} \) is the limit appearing on the right-hand side of (4.9).

In some cases, it is useful to consider functions \( g \) which depend on the infinite path \((V_{T_{u}^{A}}, V_{T_{u}^{A}+1}, \ldots)\), or to consider functions \( g \) which need not be bounded. Moreover, it is also useful to have uniform upper bounds. In these situations, a variant of the above proposition is useful.

Proposition 4.5. Suppose that \( g : (\mathbb{R}^{d}_{+})^{\infty} \to [0, \infty) \) is a nonnegative measurable function, and set
\[
\mathfrak{G}_{u} = \nu^{a}\alpha(\bar{\nu})\left(\frac{|V_{T_{u}^{A}}|}{u}\right)^{-\alpha} \mathbb{E}\left[g\left(\left\{(\frac{V_{T_{u}^{A}+k}}{u})_{k \geq 0}\right\}\big| \mathfrak{F}_{T_{u}^{A}}\right].
\]
Further assume that for some finite constant $B$ and some $\mathcal{U} \geq 0$,
\[
\sup_{u \geq \mathcal{U}} \hat{\Theta}_u \leq B \quad \mathbb{P}^\alpha\text{-a.s.} \tag{4.17}
\]

Then for any bounded set $F \subset \mathbb{R}^d_+ \setminus \{0\}$, there exists a finite constant $L$, not depending on $B$, such that
\[
0 \leq \sup_{u \geq \mathcal{U}} \sup_{v \in F} u^\alpha \mathbb{E} \left[ g \left( \frac{V_{T_u+k}}{u} \right)_{k \geq 0} \mathbf{1}_{\{T_u^\alpha < \tau\}} \bigg| V_0 = v \right] \leq BL. \tag{4.18}
\]

Moreover, if (4.17) holds and $\limsup_{u \to \infty} \hat{\Theta}_u = 0 \mathbb{P}^\alpha$-a.s., then we also have that
\[
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ g \left( \frac{V_{T_u+k}}{u} \right)_{k \geq 0} \mathbf{1}_{\{T_u^\alpha < \tau\}} \bigg| V_0 = v \right] = 0. \tag{4.19}
\]

**Proof.** Repeating the argument in the proof of Proposition 4.1 leading to (4.15), we obtain that
\[
0 \leq \sup_{u \geq \mathcal{U}} \sup_{v \in F} u^\alpha \mathbb{E} \left[ g \left( \frac{V_{T_u+k}}{u} \right)_{k \geq 0} \mathbf{1}_{\{T_u^\alpha < \tau\}} \bigg| V_0 = v \right] \leq Br_\alpha(\bar{v}) \sup_{v \in F} \mathbb{E}^\alpha \left[ \sup_{n \in \mathbb{N}} \left| Z_n \right| \right] \sup_{n \in \mathbb{N}} \mathbf{1}_{\{n \leq \tau\}}
\]
which is finite by Lemma 3.6 (iii) and the boundedness of $r_\alpha$. The boundedness of $\hat{\Theta}_u$ further allows us to use the dominated convergence theorem in order to deduce (4.19) from (4.15). \[ \square \]

The remainder of this section is devoted to the proof of several results that are needed in order to establish Theorem 2.5 in the subsequent section. Therefore, we now restrict our attention to the case where $A = \{v \in \mathbb{R}^d_+ : |v| > 1\}$; and thus, $T_u^A = T_u$, $d_A = 1$, $r_u^A = r_\alpha$, and $g^A = g$.

Recall that in Proposition 4.1, we studied the behavior of $(V_n)$ over paths of finite length. Now suppose that we replace the function $g$ in that proposition with a function of the form $\sum_{k=0}^{\tau-1} h(V_{T_u+k}/u)$. In the next lemma, we show that it is the path behavior over finite time intervals of the form $[T_u, T_u+m]$ which plays the determining role. Additionally, we establish a technical result, given in (4.22) below, stating that if one computes the path behavior over $[T_u, T_u+m]$ when $T_u - \tau < m$, then the effect of these additional terms is, roughly speaking, negligible.

**Lemma 4.6.** Let $h$ be a bounded measurable function such that $h(x) = 0$ for all $x \in B_\delta(0)$, for some $\delta > 0$. Then for all $v \in \mathbb{R}^d_+ \setminus \{0\}$ and all $m \in \mathbb{N}$,
\[
\lim_{m \to \infty} \lim_{u \to \infty} u^\alpha \mathbb{E} \left[ \sum_{k=m}^{\tau-1-T_u} h \left( \frac{V_{T_u+k}}{u} \right) \mathbf{1}_{\{T_u+m < \tau\}} \bigg| V_0 = v \right] = 0. \tag{4.20}
\]

Moreover, if $F \subset \mathbb{R}^d_+ \setminus \{0\}$ is bounded, then by summing over all terms in the interval $[T_u, \tau)$, we obtain that
\[
\limsup_{u \to \infty} \sup_{v \in F} u^\alpha \mathbb{E} \left[ \sum_{i=0}^{\tau-1} h \left( \frac{V_i}{u} \right) \mathbf{1}_{\{|V_i| > u\}} \bigg| V_0 = v \right] < \infty. \tag{4.21}
\]

Furthermore, for all $v \in \mathbb{R}^d_+ \setminus \{0\}$,
\[
\lim_{m \to \infty} \lim_{u \to \infty} u^\alpha \mathbb{E} \left[ \sum_{i=\tau}^{T_u+m} h \left( \frac{V_i}{u} \right) \mathbf{1}_{\{T_u < \tau \leq T_u+m\}} \bigg| V_0 = v \right] = 0. \tag{4.22}
\]

**Proof.** **Step 1.** First we establish (4.20). By Eq. (4.18) in Proposition 4.5, it suffices to prove that
\[
\sup_{u \geq \mathcal{U}} \hat{\Theta}_u := \sup_{u \geq \mathcal{U}} \frac{1}{r_\alpha(X_{T_u})} \left( \frac{|V_{T_u}|}{u} \right)^{-\alpha} \mathbb{E} \left[ \sum_{k=m}^{\tau-1-T_u} h \left( \frac{V_{T_u+k}}{u} \right) \mathbf{1}_{\{T_u+m < \tau\}} \bigg| \mathcal{F}_{T_u} \right] \leq B(m, \mathcal{U})
\]

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for a sequence \( \{B(m, \mathcal{W})\} \) which tends to zero as we first let \( \mathcal{W} \to \infty \) and then let \( m \to \infty \). By employing the Markov property and the boundedness of \( r_{\alpha} \), we see that it is enough to show that, for a suitable sequence \( B(m, \mathcal{W}) \),

\[
\sup_{u \geq \mathcal{W}} \sup_{v: |v| \geq u} H_u(v) := \sup_{u \geq \mathcal{W}} \sup_{v: |v| \geq u} E \left[ \left( \frac{|V_0|}{u} \right)^{-\alpha} \sum_{k=m}^{\tau - 1} h \left( \frac{V_k}{u} \right) 1_{\{\tau > \tau_k\}} \right] \leq B(m, \mathcal{W}). \tag{4.23}
\]

Now let \( \mathbb{D}^\dagger = \{v \in \mathbb{R}^d_+ : |v| \leq L\} \) be defined as in Lemma 3.1, and set \( \tau^\dagger := \inf \{n \in \mathbb{N}_+ : V_n \in \mathbb{D}^\dagger\} \).

Recall that \( h(x) = 0 \) for all \( x \in B\delta(0) \). Hence, for \( 0 < \theta < \min\{1, \alpha\} \) and \( |v| > u \), we have

\[
H_u(v) \leq |h|_{\infty} \left( \frac{|V_0|}{u} \right)^{-\alpha} \left( \sum_{k=m}^{\tau - 1} 1_{\{|V_k| > \delta u\}} + \sum_{k=\tau}^{\tau - 1} 1_{\{|V_k| > \delta u\}} \right) \leq |h|_{\infty} \left( \frac{|v|}{u} \right)^{-\alpha} \sum_{k=m}^{\tau - 1} (\delta u)^{-\theta} E_v[|V_k|^\theta 1_{\{|\tau^\dagger > k\}}] + |h|_{\infty} \sup_{w \in \mathbb{D}^\dagger} \sup_{w \in \mathbb{D}^\dagger} \left( \sum_{k=m}^{\tau - 1} 1_{\{|V_k| > \delta u\}} \right).
\]

The first sum can be estimated further by employing Lemma 3.1:

\[
\sup_{v: |v| \geq u} \left( \frac{|v|}{u} \right)^{-\alpha} \sum_{k=m}^{\tau - 1} (\delta u)^{-\theta} E_v[|V_k|^\theta 1_{\{|\tau^\dagger > k\}}] \leq \frac{B}{\delta^\theta} \left( \sup_{v: |v| \geq u} \left( \frac{|v|}{u} \right)^{\theta - \alpha} \right) \frac{\tau^m}{1 - \tau} = \frac{B}{\delta^\theta} \frac{\tau^m}{1 - \tau},
\]

and the last term tends to 0 as \( m \to \infty \). For the second term, note that (3.8) implies that \( \sup_{w \in \mathbb{D}^\dagger} E[\tau | V_0 = w] < \infty \). Hence we can apply a dominated convergence argument to infer that

\[
\sup_{u \geq \mathcal{W}} \sup_{w \in \mathbb{D}^\dagger} \mathbb{E}_w \left[ \sum_{k=0}^{\tau - 1} 1_{\{|V_k| > \delta u\}} \right] \leq \sup_{w \in \mathbb{D}^\dagger} \mathbb{E}_w \left[ \sum_{k=0}^{\tau - 1} 1_{\{|V_k| > \delta u\}} \right] \to 0 \text{ as } \mathcal{W} \to \infty.
\]

Combining these estimates, we have established (4.23), and (4.20) follows.

Finally, (4.21) is a direct consequence of the above estimates for \( m = 0 \) combined with (4.18).

STEP 2. Turning to (4.22), we now apply the second part of Proposition 4.5. Using that \( h = 0 \) on \( B_d(0) \), it is now sufficient to show that for any fixed \( m \in \mathbb{N}, \)

\[
\bar{\mathcal{G}}_u := \frac{1}{r_{\alpha}(X_{\tau_u})} \left( \frac{|V_{\tau_u}|}{u} \right)^{-\alpha} E \left[ \sum_{i=\tau}^{\tau + m} h \left( \frac{V_i}{u} \right) 1_{|V_i| > \delta u} 1_{T_u < \tau T_u + m} \right] \mathcal{F}_{\tau_u} \tag{4.24}
\]

is bounded uniformly in \( u \) and tends to zero \( \mathbb{P}^\alpha \)-a.s. as \( u \to \infty \). The prefactors are bounded, and thus it suffices to estimate

\[
E \left[ \sum_{i=\tau}^{\tau + m} h \left( \frac{V_i}{u} \right) 1_{|V_i| > \delta u} 1_{T_u < \tau T_u + m} \right] \mathcal{F}_{\tau_u} \leq |h|_{\infty} E \left[ \sum_{i=\tau}^{\tau + m} 1_{|V_i| > \delta u} 1_{T_u < \tau} \right] \mathcal{F}_{\tau_u} \mathcal{F}_{\tau_u} \tag{4.25}
\]

Now let \( \theta \in (0, \alpha) \). Then for all \( k = 0, \ldots, m, \)

\[
\mathbb{P} \left( |V_{\tau + k}| > \delta u | V_{\tau} = v \right) \leq \sup_{v \in \mathbb{D}} \mathbb{P} \left( |V_k| > \delta u | V_0 = v \right) \leq \sup_{v \in \mathbb{D}} (\delta u)^{-\theta} E \left[ |V_k|^\theta | V_0 = v \right] \leq \delta^{-\theta} u^{-\theta} \left( \sup_{v \in \mathbb{D}} E[\|M_k \cdots M_1\|^\theta] \cdot |v|^\theta + \sum_{j=1}^k E[|M_k \cdots M_{j+1}Q_j|^\theta] \right) \leq B_1 u^{-\theta}
\]

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for some constant \( B_1 = B_1(m) < \infty \). Substituting the last estimate into (4.25) and then (4.24), we obtain that for some finite constant \( B_2 \),

\[
\Phi_u \leq B_2 m u^{-\theta} \downarrow 0 \quad \text{as} \quad u \to \infty.
\]

Thus, by Proposition 4.5, we have for all \( m \in \mathbb{N} \) that

\[
\lim_{u \to \infty} u^{\alpha} E \left[ \tau \left( \sum_{i=0}^{m} f \left( \frac{V_i}{u} \right) 1 \{ |V_i| \geq u \} \right) V_0 = v \right] = 0
\]

and (4.22) follows.

Now by combining Corollary 4.3 and Lemma 4.6, we obtain the following result. In this lemma, the function \( f \) will correspond to that function appearing in the statement of Theorem 2.5.

**Lemma 4.7.** Assume Hypotheses \((H_1)\) and \((H_2)\) are satisfied. Let \( \theta \leq \min\{1, \alpha\} \) and let \( f \) be a nonnegative bounded \( \theta \)-Hölder continuous function. Then for all \( v \in \mathbb{R}_+^d \setminus \{0\} \),

\[
\lim_{u \to \infty} u^{\alpha} E \left[ \tau \left( \sum_{i=0}^{m} f \left( \frac{V_i}{u} \right) 1 \{ |V_i| \geq u \} \right) V_0 = v \right] = r_\alpha(\nu) E_{\nu} \left[ \alpha \left( |Z| \right) 1 \{ \tau = \infty \} \right] \lim_{m \to \infty} \int_{S_{d-1} \times \mathbb{R}_+} E^v \left[ \sum_{i=0}^{m} F(X_i, S_i + s) \right] g(dx, ds), \tag{4.26}
\]

where \( F(x, s) := (e^{-\alpha s} f(e^s x) / r_\alpha(x)) 1_{\{0, \infty\}}(s) \).

**Proof.** Let \( g(v) := f(v) 1_{\{|v| \geq 1\}} \), and note that \( g \) is an almost \( \theta \)-Hölder-continuous function. For \( m \in \mathbb{N}_+ \), consider the decomposition

\[
\begin{align*}
&u^{\alpha} E_v \left[ \tau \left( \sum_{i=0}^{m} f \left( \frac{V_i}{u} \right) 1 \{ |V_i| \geq u \} \right) V_0 = v \right] = u^{\alpha} E_v \left[ \sum_{i=0}^{m} f \left( \frac{V_i}{u} \right) 1 \{ |V_i| \geq u \} 1_{\{T_u < \tau\}} \right] \\
&\quad = u^{\alpha} \sum_{i=0}^{m} E_v \left[ g \left( \frac{V_{i+1}}{u} \right) 1_{\{T_u < \tau\}} \right] + u^{\alpha} E_v \left[ \sum_{i=m+1}^{\tau-1} g \left( \frac{V_{i+1}}{u} \right) 1_{\{T_u + m < \tau\}} \right] \\
&\quad \quad - u^{\alpha} E_v \left[ \sum_{i=\tau}^{\tau + m} g \left( \frac{V_i}{u} \right) 1_{\{T_u < \tau \leq T_u + m\}} \right]. \tag{4.27}
\end{align*}
\]

On the right-hand side of (4.27), the last two terms tend to zero by Lemma 4.6 when taking first the limit \( u \to \infty \) and then \( m \to \infty \). Next, by Corollary 4.3, we obtain that

\[
\begin{align*}
&\lim_{u \to \infty} u^{\alpha} \sum_{i=0}^{m} E \left[ g \left( \frac{V_{i+1}}{u} \right) 1_{\{T_u < \tau\}} \right] 1_{\tau = \infty} \int_{S_{d-1} \times \mathbb{R}_+} E^v \left[ \frac{e^{-\alpha(S_i + s)}}{r_\alpha(S_i)} g(e^{S_i + s} X_i) \right] q(dx, ds) \\
&\quad = r_\alpha(\nu) E_{\nu} \left[ \alpha \left( |Z| \right) 1 \{ \tau = \infty \} \right] \int_{S_{d-1} \times \mathbb{R}_+} E^v \left[ \sum_{i=0}^{m} \frac{e^{-\alpha(S_i + s)}}{r_\alpha(S_i)} f(e^{S_i + s} X_i) 1_{\{S_i + s \geq 0\}} \right] q(dx, ds),
\end{align*}
\]

and the assertion follows by letting \( m \to \infty \). \( \square \)

To bring the limit (as \( m \to \infty \)) inside the sum in (4.26), we first need to introduce the definition of a multivariate directly Riemann integrable function.
Definition 4.8. A measurable function $F : \mathbb{S}_+^{d-1} \times \mathbb{R} \to \mathbb{R}$ is called directly Riemann integrable if for all $x \in \mathbb{S}_+^{d-1}$, the function $t \mapsto F(x, t)$ is continuous a.e. with respect to Lebesgue measure on $\mathbb{R}$, and
\[
\sum_{l=-\infty}^{\infty} \sup \left\{ |F(x, t)| : x \in \mathbb{S}_+^{d-1}, t \in [l, l+1] \right\} < \infty. \tag{4.28}
\]

In particular, $F$ directly Riemann integrable implies that
\[
\sup_{x \in \mathbb{S}_+^{d-1}} \sup_{s \in \mathbb{R}} \mathbb{E}_x^\alpha \left[ \sum_{i=0}^{\tau-1} f \left( \frac{V_i}{u} \right) 1_{\{|V_i| \geq u\}} |V_0 = v\right] = r_\alpha(\overline{r}) \mathbb{E}_x^\alpha \left[ |Z|^\alpha 1_{\{|\tau| = \infty\}} \right] \int \mathbb{E}_x^\alpha \left[ \sum_{i=0}^{\infty} F(X_i, S_i + s) \right] g(dx, ds). \tag{4.30}
\]

Proof. Since $r_\alpha$ is bounded from below, it follows that for some positive constant $b$,
\[
\overline{F}(s) := \sup_{x \in \mathbb{S}_+^{d-1}} |F(x, s)| \leq \frac{1}{b} |f|_\infty e^{-\alpha s} 1_{[0, \infty)}(s).
\]

Since the right-hand side is a decreasing integrable function, we conclude that $\overline{F}$ is (univariate) directly Riemann integrable. Since $\overline{F}$ is obtained from $F$ by taking the supremum over all $x \in \mathbb{S}_+^{d-1}$, it follows immediately from Definition 4.8 that $F$ is (multivariate) Riemann integrable. Then by (4.29), we can use a dominated convergence argument to interchange $\lim_{m \to \infty}$ with the integration in (4.26).

5 Proof of Theorem 2.5

In this section, we provide the proof of Theorem 2.5, first under the additional hypothesis $(H_3)$ of Section 3, which is then removed by approximating $\{V_n\}$ from above and below by smoothed processes for which $(H_3)$ is satisfied.

To establish Theorem 2.5, we apply Proposition 4.9 directly, except that we must identify the integral in (4.30). Note that if we were in the setting of classical random walk—where $\{\xi_i\}$ is an i.i.d. sequence of random variables and $S_n = \sum_{i=1}^{n} \xi_i + S_0$—and if we were to consider $F(s) = 1_{[0, \ell]}(s)$ in (4.30), then the integral in (4.30) would represent the renewal function, with $S_0$ having the stationary excess distribution $\rho$. It is known (see e.g. Proposition 1.0, Eq. (1.4), in Thorisson (1987)) that for $S_0 \sim \rho$, the renewal function is equal to $t/E[\xi]$. In other words, if $S_0 \sim \rho$, then the renewal measure $\sum_{i=0}^{\infty} \mathbb{P}(S_i \in \cdot)$, restricted to $[0, \infty)$, is equal to $1/E[\xi]$ multiplied by Lebesgue measure on $[0, \infty)$.

Our present objective is to extend this identity into our Markovian framework. Here, $E[\xi]$ must be replaced by the drift of $S_1$ under $\mathbb{P}_\alpha$, which is $\lambda'(\alpha)$, and instead of Lebesgue measure on $\mathbb{R}_+$, we expect to obtain the limiting measure from Kesten’s renewal theorem, namely the measure $\eta_\alpha \otimes I$ on $\mathbb{S}_+^{d-1} \times \mathbb{R}_+$, where $I$ denotes Lebesgue measure.

Lemma 5.1. Let $g : \mathbb{S}_+^{d-1} \times \mathbb{R} \to \mathbb{R}$ be a directly Riemann integrable function. Then
\[
\int_{\mathbb{S}_+^{d-1} \times \mathbb{R}_+} \mathbb{E}_x^\alpha \left[ \sum_{i=0}^{\infty} g(X_i, S_i + s) 1_{\{s+s \geq 0\}} \right] g(dx, ds) = \frac{1}{\lambda'(\alpha)} \int_{\mathbb{S}_+^{d-1} \times \mathbb{R}_+} g(x, s) \eta_\alpha(dx) ds. \tag{5.1}
\]
Before proving this result, we start by recalling the standard extension of \(\{(X_n, S_n) : n = 0, 1, \ldots\}\) to a doubly-infinite stationary process, which, in particular, may be used to identify the measure \(\varrho\) appearing on the left-hand side of (5.1). For this purpose, let \(\xi_n := S_n - S_{n-1}\) and recall that \(\{\{X_n, \xi_n\} : n = 1, 2, \ldots\}\) is stationary under \(\mathbb{F}^\psi\); cf. (2.5). Now let \(\{X_n^{\psi}, \xi_n^{\psi}\} : -\infty < n < \infty\) be the two-sided extension of this stationary sequence \(\{\{X_n, \xi_n\} : n = 1, 2, \ldots\}\), defined on a probability space \((\Omega^\psi, \mathbb{F}^\psi, \mathbb{P}^\psi)\). Then \(\{\{X_n^{\psi}, \xi_n^{\psi}\} : -\infty < n < \infty\}\) is a stationary, doubly-infinite Markov chain; and for each \(k \in \mathbb{Z}\),
\[
\mathbb{P}^\psi\left(\{X_{k+n}^{\psi}, \xi_{k+n}^{\psi}\}_{n \geq 0} \in \cdot\right) = \mathbb{P}^\alpha\left(\{X_n, \xi_n\}_{n \geq 0} \in \cdot\right).
\]
Further define
\[
S_n^{\psi} := \begin{cases} 
\sum_{k=1}^{n} \xi_k^{\psi}, & n > 0, \\
0, & n = 0, \\
-\sum_{k=n+1}^{0} \xi_k^{\psi}, & n < 0.
\end{cases}
\]
Then for all \(k \in \mathbb{Z}\), \(S_k^{\psi} - S_{k+1}^{\psi} = \xi_k^{\psi}\). Next introduce the ladder indices for \(\{\{X_n^{\psi}, S_n^{\psi}\}\}\), namely,
\[
\begin{align*}
\zeta_0 & := \sup\{n \leq 0 : S_n^{\psi} > \sup_{j < n} S_j^{\psi}\}; \\
\zeta_{k+1} & := \inf\{n > \zeta_k^{\psi} : S_n^{\psi} > S_{\zeta_k^{\psi}}^{\psi}\}, \quad k = 0, 1, \ldots;
\end{align*}
\]
and the ladder indices for \(\{\{X_n, S_n\}\}\), namely,
\[
\zeta_0 := 0; \quad \zeta_{k+1} := \inf\{n > \zeta_k^{\psi} : S_n > S_{\zeta_k}^{\psi}\}, \quad k = 0, 1, \ldots \tag{5.2}
\]
In particular, \(\zeta_1 = \inf\{n > 0 : S_n > 0\}\). Also define the measure \(\psi\) on \(\mathbb{S}_+^{d-1}\) by setting
\[
\psi(A) = \mathbb{P}^\psi\left(\zeta_0^{\psi} = 0, X_0^{\psi} \in A\right).
\]
Then by Kesten’s renewal theorem,
\[
\varrho(E \times \Gamma) = \frac{1}{\lambda(\alpha)} \int_\Gamma \mathbb{P}_0^\psi (X_{\zeta_1}^{\psi} \in E, S_{\zeta_1} > s) \, ds, \quad E \subset \mathbb{S}_+^{d-1}, \quad \Gamma \in \mathbb{S}_+; \tag{5.3}
\]
see Eqs. (1.16) and (3.10) in Kesten (1974).

**Proof of Lemma 5.1.** If the function \(g\) takes both positive and negative values, then as the left- and right-hand sides of (5.1) are finite (cf. (4.29)), we may split \(g\) into its positive and negative parts, applying the result separately to each of these parts. Thus, for the remainder of the proof, we will assume without loss of generality that \(g\) is nonnegative.

**Step 1.** We start by applying Lemma 3 of Kesten (1974) to obtain an expression analogous to (5.1), but with respect to the positive ladder heights. Set \(h_t(x, s) = g(x, t - s)\) and
\[
\mathcal{A}(x, E, t) = \sum_{k=0}^{\infty} \mathbb{P}_x\left(X_{\zeta_k}^{\psi} \in E, S_{\zeta_k}^{\psi} \in [0, t]\right), \quad x \in \mathbb{S}_+^{d-1}, \quad E \in \mathcal{B}(\mathbb{S}_+^{d-1});
\]
cf. Kesten (1974), Eq. (3.26). Then it follows from these definitions that
\[
\int_{x \in \mathbb{S}_+^{d-1}, \, 0 \leq r \leq t - s} h_t(y, t - r - s) \mathcal{A}(x, dy, dr) = \mathbb{E}_x^\alpha \left[ \sum_{k=0}^{\infty} g(X_{\zeta_k}^{\psi}, S_{\zeta_k}^{\psi} + s) \mathbf{1}_{\{S_{\zeta_k}^{\psi} + s \geq 0\}} \right].
\]
Hence by Lemma 3 of Kesten (1974) and (5.3), we obtain upon letting \(t \uparrow \infty\) that
\[
\int_{x \in \mathbb{S}_+^{d-1} \times \mathbb{R}_+} \mathbb{E}_x^{\alpha} \left[ \sum_{k=0}^{\infty} g(X_{\zeta_k}^{\psi}, S_{\zeta_k}^{\psi} + s) \mathbf{1}_{\{S_{\zeta_k}^{\psi} + s \geq 0\}} \right] g(dx, ds) = \frac{1}{\lambda(\alpha)} \int_{\mathbb{S}_+^{d-1}} \psi(dx) \int_0^t h_t(x, s) ds
\]
\[
= \frac{1}{\lambda(\alpha)} \int_{\mathbb{S}_+^{d-1} \times \mathbb{R}_+} g(x, s) \psi(dx) ds. \tag{5.4}
\]
STEP 2. We would now like to extend (5.4) so that the sum on the left-hand side is taken over all \((X_n, S_n)\), rather than those values corresponding to the ladder heights. Using a cyclic decomposition, observe that for any \((x, s) \in S_+^{d-1} \times \mathbb{R}_+\),
\[
\mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\infty} g(X_n, S_n + s) 1_{\{S_n + s \geq 0\}} \right] = \mathbb{E}_x^\alpha \left[ \sum_{k=0}^{\infty} G(X_{\varsigma_k}, S_{\varsigma_k} + s) 1_{\{S_{\varsigma_k} + s \geq 0\}} \right],
\]
where
\[
G(y, t) := \mathbb{E}_y^\alpha \left[ \sum_{n=0}^{\varsigma_1 - 1} g(X_n, S_n + t) 1_{\{S_n + t \geq 0\}} \right], \quad t \in \mathbb{R}.
\]
Note that \(g\) directly Riemann integrable implies that so is \(G\). Now apply (5.4) with \(G\) in place of \(g\) to obtain that
\[
\int_{S_+^{d-1} \times \mathbb{R}_+} \mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\infty} g(X_n, S_n + s) 1_{\{S_n + s \geq 0\}} \right] g(dx, ds) = \frac{1}{\lambda'(\alpha)} \int_{S_+^{d-1} \times \mathbb{R}_+} G(x, s) \psi(dx) ds
\]
\[
= \frac{1}{\lambda'(\alpha)} \int_{S_+^{d-1} \times \mathbb{R}_+} \psi(dx) \mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\varsigma_1 - 1} g(X_n, S_n + s) 1_{\{S_n + s \geq 0\}} ds \right]
\]
where the last step follows by observing that \(S_n < 0\) prior to the first ascending ladder height, which implies that \(\int_{\mathbb{R}_+} g(X_n, S_n + s) 1_{\{S_n + s \geq 0\}} ds = \int_{\mathbb{R}_+} g(X_n, s) ds\).

STEP 3. To establish the lemma, it remains to show that
\[
\int_{S_+^{d-1}} \psi(dx) \mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\varsigma_1 - 1} g(X_n, s) \right] = \int_{S_+^{d-1}} g(x, s) \eta_\alpha(dx).
\]
Approximating both the positive and negative parts of \(x \mapsto g(x, s)\) by simple functions, it is sufficient to show that
\[
\int_{S_+^{d-1}} \psi(dx) \mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\varsigma_1 - 1} 1_{\{X_n \in E\}} \right] = \eta_\alpha(E), \quad E \in \mathcal{B}(S_+^{d-1}).
\]
To this end, observe using the definition of \(\psi\) that
\[
\int_{S_+^{d-1}} \psi(dx) \mathbb{E}_x^\alpha \left[ \sum_{n=0}^{\varsigma_1 - 1} 1_{\{X_n \in E\}} \right] = \sum_{l=0}^{\infty} \int_{S_+^{d-1}} \mathbb{P}^2 \left( \sup_{j < 0} S_j^2 < 0, X_0^l \in E \right) \mathbb{P}_2^\alpha(X_l \in E, \varsigma_1 > l)
\]
\[
= \sum_{l=0}^{\infty} \mathbb{P}^2 \left( \sup_{j < 0} S_j^2 < 0, \sup_{1 \leq j \leq l} S_j^2 \leq 0, X_l^l \in E \right)
\]
\[
= \sum_{l=0}^{\infty} \mathbb{P}^2 \left( -l, X_0^l \in E \right) = \mathbb{P}^2 \left( -\infty < \varsigma_0^2, X_0^l \in E \right),
\]
where in the last line, we have used the stationarity of the process \((X_t^l, \varsigma_0^2) : -\infty < n < \infty\) and the fact that \(S_0^2 = 0\). But \(\mathbb{P}^2(-\infty < \varsigma_0^2) = 1\) (cf. Kesten (1974), Lemma 2, Eq. (3.12)), and hence
\[
\mathbb{P}^2 \left( -\infty < \varsigma_0^2, X_0^l \in E \right) = \mathbb{P}^2(X_0^l \in E) = \mathbb{P}^\alpha(X_0 \in E) = \eta_\alpha(E).
\]
Then (5.8) is obtained by substituting (5.10) into (5.9).

We now establish Theorem 2.5 under the additional Hypothesis \((H_3)\) of Section 3. This assumption will later be removed using a smoothing argument.
Proposition 5.2. Assume that Hypotheses (H1), (H2) and (H3) are satisfied, and suppose that $\mathbb{D} \in \mathcal{B}(\mathbb{R}_+^d \setminus \{0\})$ is bounded and $\pi(\mathbb{D}) > 0$. Then for any $f \in C_0(\mathbb{R}_+^d \setminus \{0\})$,

$$
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ f \left( \frac{V}{u} \right) \right] = \frac{C}{\lambda'(\alpha)} \int_{S^{d-1} \times \mathbb{R}} e^{-\alpha s} f(e^x) l_\alpha(dx)ds,
$$

(5.11)

where $C$ is given as in (2.15). Equivalently, we have the weak convergence

$$
\lim_{u \to \infty} u^\alpha \mathbb{P} \left( |V| > tu, \frac{V}{|V|} \in \cdot \right) = \frac{C}{\alpha \lambda'(\alpha)} t^{-\alpha} l_\alpha(\cdot), \quad \text{for all } t > 0.
$$

(5.12)

Proof. We first prove the result under the additional assumptions that $f$ is almost $\theta$-Hölder continuous (as defined in Eq. (4.3)) for $\theta \leq \min\{1, \alpha\}$.

**Step 1.** First assume that $f(x) = \hat{f}(x) 1_{\{|x| \geq 1\}}$ for a $\theta$-Hölder continuous function $\hat{f}$ (i.e., $\delta = 1$ in Eq. (4.3)). Since (H3) is satisfied, it follows from Lemma 3.5 that

$$
\mathbb{E} \left[ \hat{f} \left( \frac{V}{u} \right) 1_{\{|V| \geq u\}} \right] = \frac{1}{E_{\mathbb{D}}[\tau]} \int_{\mathbb{D}} \mathbb{E} \left( \sum_{i=0}^{\tau-1} \hat{f} \left( \frac{V_i}{u} \right) 1_{\{|V_i| \geq u\}} \bigg| V_0 = v \right) \pi_{\mathbb{D}}(dv),
$$

(5.13)

where $\tau$ denotes the first return time of $\{V_n\}$ to $\mathbb{D}$.

Now apply Proposition 4.9 and (5.13) (separately to the positive and negative parts of $\hat{f}$) to obtain that

$$
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ \hat{f} \left( \frac{V}{u} \right) 1_{\{|V| \geq u\}} \right] = \frac{1}{E_{\mathbb{D}}[\tau]} \int_{\mathbb{D}} \mathbb{E} \left( \sum_{i=0}^{\tau-1} \hat{f} \left( \frac{V_i}{u} \right) 1_{\{|V_i| \geq u\}} \bigg| V_0 = v \right) \pi_{\mathbb{D}}(dv),
$$

(5.14)

where

$$
F(x, s) = (e^{-\alpha s} f(e^x)/r_\alpha(x)) 1_{\{0, \infty\}}(\cdot) \quad \text{(since } 1_{\{|x| \geq 1\} \text{ can be dropped in the last integral in (5.14)).}
$$

We remark that Proposition 4.9 is conditional on $V_0 = v$. To extend this result so that it holds conditional on $V_0 \sim \pi_{\mathbb{D}}$, we have applied a dominated convergence argument together with the bound provided by (4.21) of Lemma 4.6. Next observe by Lemma 5.1 that

$$
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ \hat{f} \left( \frac{V}{u} \right) 1_{\{|V| \geq u\}} \right] = \frac{1}{\lambda'(\alpha)} \int_{S^{d-1} \times \mathbb{R}_+} E_x^\alpha \left[ \sum_{i=0}^{\tau} F(X_i, S_i + s) \right] \mathbb{E}_{x}^\alpha \left[ \sum_{i=0}^{\tau} F(X_i, S_i + s) \right] \pi_{\mathbb{D}}(dv),
$$

(5.15)

using that $\eta_\alpha(dx) = r_\alpha(x)\eta_\alpha(dx)$ (cf. (2.4) and the discussion given there).

Moreover, by Lemma 3.3, we have that $\pi(\mathbb{D}) = (E_{\pi_{\mathbb{D}}}[\tau])^{-1}$, and recall that $\pi_{\mathbb{D}}(\cdot) = \pi(\cdot \cap \mathbb{D})/\pi(\mathbb{D})$. Hence, by applying Lemma 3.6 (ii), we obtain that

$$
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ f \left( \frac{V}{u} \right) \right] = \frac{1}{E_{\pi_{\mathbb{D}}}[\tau]} \int_{\mathbb{D}} \mathbb{E} \left( \sum_{i=0}^{\tau} F(X_i, S_i + s) \right) \pi_{\mathbb{D}}(dv) = \frac{1}{E_{\pi_{\mathbb{D}}}[\tau]} \int_{\mathbb{D}} \mathbb{E} \left( \sum_{i=0}^{\tau} F(X_i, S_i + s) \right) \pi_{\mathbb{D}}(dv) = C,
$$

(5.16)

where $C$ is given as in (2.15). Then (5.14), (5.15), and (5.16) imply that

$$
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ f \left( \frac{V}{u} \right) 1_{\{|V| \geq u\}} \right] = \frac{1}{\lambda'(\alpha)} \int_{S^{d-1} \times \mathbb{R}_+} e^{-\alpha s} f(e^x) l_\alpha(dx)ds
$$

(5.17)

for any bounded, almost $\theta$-Hölder continuous function $f$ of the form $f(x) = \hat{f}(x) 1_{\{|x| \geq 1\}}$, where $\hat{f}$ is $\theta$-Hölder continuous.

**Step 2.** Now suppose that $f$ is a bounded, almost $\theta$-Hölder continuous function with support in $(B_0^d(0))^c$ for some $\delta > 0$, and define $f(x) = \delta^{-\alpha} f(\delta v)$. Then $\hat{f}$ is almost Hölder-continuous with support in $(B_0^d(0))^c$, and we infer from (5.17) that

$$
\mathbb{E} \left[ f \left( \frac{V}{u} \right) 1_{\{|V| \geq \delta u\}} \right] = \mathbb{E} \left[ f \left( \frac{V}{\delta u} \right) 1_{\{|V| \geq \delta u\}} \right] \overset{u \to \infty}{\longrightarrow} \frac{C}{\lambda'(\alpha)} \int_{S^{d-1} \times \mathbb{R}_+} e^{-\alpha s} \tilde{f}(e^x) l_\alpha(dx)ds
$$

(5.18)
\[
= \frac{C}{L'(\alpha)} \int_0^\infty \int_{\mathbb{R}^d_+} e^{-\alpha s} \delta^{-\alpha} f(\delta e^s x) l_\alpha(dx) ds \\
= \frac{C}{L'(\alpha)} \int_0^\infty \int_{\mathbb{R}^d_+} e^{-\alpha r} f(e^r x) l_\alpha(dx), \quad \text{where } r = s + \log \delta.
\]

Since \( f \) vanishes on \( B_1(0) \), we may extend the outer integral to range from 0 to \( \infty \). Thus we have obtained (5.11) for almost \( \theta \)-Hölder continuous functions on \( \mathbb{R}^d_+ \setminus \{0\} \).

**Step 3.** It remains to remove the assumption that \( f \) is almost \( \theta \)-Hölder continuous, needed to apply Proposition 4.9 in the above argument. To this end, observe that for all \( r > 0 \),
\[
\Upsilon_u^{(r)} := u^\alpha \mathbb{P} \left( \frac{V}{u} \in \cdot, \frac{|V|}{u} \geq r \right)
\]
defines a family of uniformly bounded measures on \( \mathbb{R}^d_+ \setminus B_r(0) \), where the boundedness follows by employing (5.11) with \( f(x) = 1_{\{|x| \leq r\}} \), which is an almost \( \theta \)-Hölder continuous function. The Fourier characters \( x \mapsto e^{i(x,y)} \) are bounded Lipschitz continuous functions for any \( y \in \mathbb{R}^d \); then \( f_u(x) := e^{i(x,y)} 1_{\{|x| \leq r\}} \) is almost \( \theta \)-Hölder continuous for any \( \theta \leq \min\{1, \alpha\} \). Let \( \Sigma_\alpha \) be the measure on \( \mathbb{R}^d_+ \setminus \{0\} \) defined by the equation
\[
\int_{\mathbb{R}^d_+ \times \mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds = \int_{\mathbb{R}^d_+ \setminus \{0\}} f(x) \Sigma_\alpha(dx),
\]
and let \( \Sigma_\alpha^{(r)} \) denote its restriction to a measure on \( \mathbb{R}^d_+ \setminus B_r(0) \). Then, based on what we have proved so far, we may infer the convergence, for all \( y \in \mathbb{R}^d \) (considering real and imaginary part separately), of
\[
\lim_{u \to \infty} \int_{\mathbb{R}^d_+ \setminus B_r(0)} e^{i(x,y)} \Upsilon_u^{(r)}(dx) = \lim_{u \to \infty} u^\alpha \mathbb{E} \left[ e^{i(x,y)} 1_{\{|y| \leq ru\}} \right] = \frac{C}{L'(\alpha)} \int_{\mathbb{R}^d_+ \setminus B_r(0)} e^{i(x,y)} \Sigma_\alpha^{(r)}(dx);
\]
and thus, by the Lévy continuity theorem, the weak convergence \( \Upsilon_u^{(r)} \Rightarrow \frac{C}{L'(\alpha)} \Sigma_\alpha^{(r)} \), for any \( r > 0 \). Now if \( \xi \in \mathcal{G}_0(\mathbb{R}^d_+ \setminus \{0\}) \), then there exists \( r > 0 \) such that \( f \) is supported on \( (B_r(0))^c \). Hence
\[
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ f \left( \frac{V}{u} \right) \right] = \lim_{u \to \infty} \int f(x) \Upsilon_u^{(r)}(dx) = \frac{C}{L'(\alpha)} \int f(x) \Sigma_\alpha^{(r)}(dx) = \frac{C}{L'(\alpha)} \int f(x) \Sigma_\alpha(dx),
\]
i.e. (5.11) holds.

Finally, the equivalence of (5.11) to (5.12) follows from Theorem 2 in Resnick (2004).

**Smoothing.** To remove Hypothesis \((H_3)\), we employ a lower and upper approximation, where the approximating sequences are smoothed so that \((H_3)\) is satisfied by these sequences.

We begin by constructing the lower approximating sequence. First recall the statement of Theorem 2.5. Also, from this discussion in Section 2, recall the definitions
\[
\tilde{M}_n := M_{k_1} \cdots M_{k_{(n-1)+1}} \quad \text{and} \quad \hat{Q}_n := \sum_{i=k_{(n-1)+1}}^{k_n} M_{k_n} \cdots M_{i+1} Q_i, \quad n \in \mathbb{N}_+.
\]
(5.18)

Now let \( k \in \mathbb{N}_+ \) be chosen such that \((\hat{R})\) holds. Then \( \{ (\tilde{M}_n, \hat{Q}_n) \} \) is an i.i.d. sequence under \( \mathbb{P} \), and with positive probability, \( Q_1 - s I > 0 \) for some \( s > 0 \). Let \( \mathbb{B}_n = \{ \hat{Q}_n - s I > 0 \} \), and let \( \chi_{n,\varepsilon} := (-\varepsilon) \chi_n \) for a sequence \( \{ \chi_n \} \) of i.i.d. random variables which are independent of \( \{ (\tilde{M}_n, \hat{Q}_n) \} \) and have a nondegenerate absolutely continuous distribution concentrated on \([0,1]^d \) (thus, \( \chi_{n,\varepsilon} \) is concentrated on \([-\varepsilon,0]^d \)). Set \( \hat{Q}_{n,\varepsilon} := \hat{Q}_n + 1_{\mathbb{B}_n} \chi_{n,\varepsilon} \) and note that conditioned on the event \( \mathbb{B}_n \), \( \hat{Q}_{n,\varepsilon} \) has a continuous distribution function. Since the event \( \mathbb{B}_n \) occurs with positive probability, this implies that the distribution function of \( \hat{Q}_{n,\varepsilon} \) has an absolutely continuous component with respect to Lebesgue measure.
Now set
\[ V_{n,\varepsilon} = \tilde{M}_n V_{n-1,\varepsilon} + \tilde{Q}_{n,\varepsilon}, \quad n = 1, 2, \ldots; \quad V_{0,\varepsilon} = V_0. \] (5.19)
Then \( \{V_{n,\varepsilon}\} \) forms the smoothed lower sequence. Let
\[ V_{\varepsilon} := \tilde{Q}_{1,\varepsilon} + \sum_{j=1}^{\infty} \tilde{M}_j \cdots \tilde{M}_j \tilde{Q}_{j+1,\varepsilon}, \] (5.20)
and note that the law of \( V_{\varepsilon} \) is the stationary distribution of the process \( \{V_{n,\varepsilon}\} \) with law \( \pi_\varepsilon \), say.

A smoothed upper sequence is constructed analogously, now choosing \( \chi_{\varepsilon} = \varepsilon \chi_n \), so that this random variable is concentrated on the interval \([0, \varepsilon]^d\). Set \( \tilde{Q}^\varepsilon_n = \tilde{Q}_n + 1_{\mathbb{B}_n} \chi_n^\varepsilon \). Then set \( V_{\varepsilon}^\varepsilon = V_0 \) and let
\[ V_{n,\varepsilon}^\varepsilon = \tilde{M}_n V_{n-1,\varepsilon}^\varepsilon + \tilde{Q}_{n,\varepsilon}^\varepsilon, \quad n = 1, 2, \ldots; \quad V_{0,\varepsilon}^\varepsilon = \tilde{Q}_{1,\varepsilon} + \sum_{j=1}^{\infty} \tilde{M}_j \cdots \tilde{M}_j \tilde{Q}_{j+1,\varepsilon}^\varepsilon. \]

Let \( \pi_\varepsilon \) denote the distribution of \( V_{\varepsilon}^\varepsilon \).

**Remark 5.3.** At this stage, it should be emphasized that this smoothing construction only affects the random quantity \( \tilde{Q}_n \) and not \( \tilde{M}_n \), and so the function \( \Lambda \) is unchanged. Thus, in particular, the solution \( \alpha \) to the equation \( \Lambda(\alpha) = 0 \) and the corresponding invariant function \( r_\alpha \) and invariant measure \( l_\alpha \) are the same as for the unsmoothed process. Moreover, since \( \tilde{M}_1 = M_k \cdots M_1 \) and \( \lambda(\alpha) = 1 \), the factor \( \lambda'(\alpha) \) must now be replaced with \( k \lambda'(\alpha) \); cf. Lemma 2.3.

**Remark 5.4.** Observe that if \( k > 1 \) in (\( \mathcal{R} \)), then the evolution of the lower and upper smoothed sequence cannot be compared to the dynamics of the process \( \{V_n\} \), but to that of the \( k \)-step chain \( \{V_{kn} : n \in \mathbb{N}\} \), which at time \( n \) is equal to
\[ \tilde{V}_n := \tilde{M}_n \cdots \tilde{M}_1 V_0 + \sum_{j=1}^{n} \tilde{M}_n \cdots \tilde{M}_{j+1} \tilde{Q}_j. \]

We then have the sandwich inequality
\[ V_{n,\varepsilon} \leq \tilde{V}_n \leq V_{n,\varepsilon}^\varepsilon, \quad \text{where} \quad \tilde{V}_n = V_{kn}. \]

For the remainder of this section, we consider the \( k \)-step chain \( \{\tilde{V}_n\} \), defined in terms of \( \{\tilde{M}_j, \tilde{Q}_j\} \). This \( k \)-step chain has the same stationary law, but different dynamics, than the 1-step chain \( \{V_n\} \).

For any \( x \in \mathbb{S}_+^{d-1} \) and \( F \subset \mathbb{S}_+^{d-1} \), let \( d(x, F) = \inf \{|x - y| : y \in F\} \), and for a given set \( E \subset \mathbb{S}_+^{d-1} \), let
\[ E_\varepsilon = \left\{ x \in \mathbb{S}_+^{d-1} : d(x, E) \leq \frac{2\varepsilon}{s} \right\} \quad \text{and} \quad E_\varepsilon^\varepsilon = \left\{ x \in \mathbb{S}_+^{d-1} : d(x, E_\varepsilon) > \frac{2\varepsilon}{s} \right\}. \]

**Lemma 5.5.** Let \( \varepsilon > 0 \). Then under the assumptions of Theorem 2.5:

(i) The approximating sequences \( \{V_{n,\varepsilon}\}_{n \in \mathbb{N}} \) and \( \{V_{n}^\varepsilon\}_{n \in \mathbb{N}} \) each satisfy Hypothesis (H3).

(ii) We have the sandwich inequality
\[ \mathbb{P} \left( |V_\varepsilon| > tu, \frac{V_\varepsilon}{|V_\varepsilon|} \in E_\varepsilon \right) \leq \mathbb{P} \left( |V| > tu, \frac{V}{|V|} \in E \right) \leq \mathbb{P} \left( |V^\varepsilon| > tu, \frac{V^\varepsilon}{|V^\varepsilon|} \in E^\varepsilon \right). \] (5.21)

**Proof.** (i) To verify part (i) of (H3), let \( P_2 \) denote the transition kernel of the process \( \{V_{n,\varepsilon}\} \) in (5.19). Recall that \( \chi_\varepsilon \) is independent of \( \tilde{M} \) and \( \tilde{Q} \). Hence, by construction, we have that
\[ P_2(v, E) = \mathbb{P} \left( \tilde{M} v + \tilde{Q} \in E, \mathbb{B}^\varepsilon \right) + \int_{-\varepsilon,0}^d \mathbb{P} \left( \tilde{M} v + \tilde{Q} + y \in E, \mathbb{B} \right) \mathbb{P} (\chi_\varepsilon \in dy) \]
\[ = P_{1,\varepsilon}(v, E) + P_{2,\varepsilon}(v, E). \]
to the process as 

Proposition 5.2 to these sequences, yielding upper and lower bounds for $\hat{P}$ and the remaining inequality in (5.21) is established by an analogous argument.

Furthermore, by (5.22), we also have that $H$.

Hence $P$ is supported on $\{\hat{v}, \hat{P}\}$, which by assumption is smooth; thus $P_2, (v, \cdot)$ itself has a Lebesgue density for all $v \in \mathbb{R}^d$.

Since $\pi \in \cdot$, it follows that $\pi$ also has a continuous component with respect to Lebesgue measure. Hence (supp $\pi$) $\notin \emptyset$ and part (ii) of (H3) is satisfied.

The verification for the process $\{V_n\}$ is analogous.

(ii) By construction,

$$V - V_\varepsilon = -1_{B_1} \chi_{1,\varepsilon} - \sum_{k=1}^\infty \hat{M}_1 \cdots \hat{M}_k 1_{B_{k+1}^\varepsilon} \chi_{k+1,\varepsilon},$$

since $\hat{Q}_k - \hat{Q}_{k,\varepsilon} = 1_{B_k} \chi_{\varepsilon}$. [Here we define $B_k$ in the same way as $B$, but with respect to the pair $(\hat{M}_k, \hat{Q}_k)$.] Consequently, setting $\hat{M}_0$ to be equal to the identity matrix and recalling that $\chi_{\varepsilon}$ is supported on $[-\varepsilon, 0]^d$, we obtain that

$$\lim_{\varepsilon \to 0} \sum_{k=0}^\infty (\hat{M}_0 \cdots \hat{M}_k \hat{P}) 1_{B_{k+1}^\varepsilon} = \lim_{\varepsilon \to 0} \sum_{k=0}^\infty (\hat{M}_0 \cdots \hat{M}_k \hat{P}) 1_{B_{k+1}^\varepsilon} = 1.$$

Moreover,

$$\frac{1}{|V|} \leq \frac{1}{|V|} + \frac{1}{|V|} \leq 2 \frac{|V - V_\varepsilon|}{|V|} \leq 2 \frac{s}{s}.$$

This implies that

$$\frac{V}{|V|} \in E \implies d \left( \frac{V}{|V|}, E \right) > \frac{2s}{s} \implies d \left( \frac{V}{|V|}, E \right) > 0 \implies V \in E.$$

Hence $\frac{V}{|V|} \in E \implies d \left( \frac{V}{|V|}, E \right) > \frac{2s}{s} \implies d \left( \frac{V}{|V|}, E \right) > 0 \implies V \in E$.

Furthermore, by (5.22), we also have that $|V_\varepsilon| > u \implies V > u$. Consequently,

$$\mathbb{P} \left( \frac{V_\varepsilon}{|V|} > u, \frac{V_\varepsilon}{|V|} \in E \right) < \mathbb{P} \left( \frac{V}{|V|} > u, \frac{V}{|V|} \in E \right) \leq \mathbb{P} \left( \frac{V}{|V|} > u, \frac{V}{|V|} \in E \right) \leq \mathbb{P} \left( \frac{V}{|V|} > u, \frac{V}{|V|} \in E \right),$$

and the remaining inequality in (5.21) is established by an analogous argument.

Since Hypothesis (H3) is satisfied for the two approximating sequences in Lemma 5.5, it is natural to apply Proposition 5.2 to these sequences, yielding upper and lower bounds for $\mathbb{P} (|V| > u, V/|V| \in E)$ as $u \to \infty$.

Let $|\hat{Z}_\varepsilon|$ be defined in the same manner as the random variable $|Z|$ in Section 3.2, but with respect to the process $\{(\hat{M}_i, \hat{Q}_i) : i = 1, 2, \ldots\}$; namely,

$$|\hat{Z}_\varepsilon| = |v| + \sum_{i=1}^\infty \left( \frac{\hat{Y}_i, (\hat{Q}_i)^\varepsilon}{\hat{Y}_i, \hat{X}_i} \right) \left( \frac{\hat{Q}_i, \varepsilon}{\hat{M}_i \cdots \hat{M}_i v} \right) \mathbb{P}^{\varepsilon}_{\hat{A}} \text{ a.s.},$$

where

$$\hat{Y}_i := \lim_{n \to \infty} \left( \hat{M}_i \cdots \hat{M}_n \hat{P} \right) \sim, \quad n = 1, 2, \ldots,$$

and $\hat{X}_i = \left( \hat{M}_i \cdots \hat{M}_i v \right)^\sim$. Then, with $E = S^{d-1}$, we obtain by Proposition 5.2, Lemma 5.5, and Remark 5.3 that

$$\frac{C_\varepsilon}{\alpha k \lambda_\varepsilon} \leq \liminf_{u \to \infty} \mathbb{P} (|V| > u) \leq \limsup_{u \to \infty} \mathbb{P} (|V| > u) \leq \frac{C_\varepsilon}{\alpha k \lambda_\varepsilon},$$

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where, in view of Lemma 3.6 (ii),

\[
C_\varepsilon = \int_{\mathbb{D}} r_\alpha (v) E^{\varepsilon}_\delta \left[ \hat{Z}_\varepsilon^\alpha 1_{\{r_\varepsilon = \infty\}} \right] \pi_\varepsilon (dv) \quad \text{and} \quad C^c = \int_{\mathbb{D}} r_\alpha (v) E^{\varepsilon}_\delta \left[ \hat{Z}_\varepsilon^\alpha 1_{\{r_\varepsilon = \infty\}} \right] \pi^c (dv).
\]

In what follows, we will generally write \( \tau \equiv \tau (\mathbb{D}) \) to emphasize the dependence of this quantity on the choice of \( \mathbb{D} \). However, it is important to observe from Proposition 5.2 that \( C_\varepsilon \) and \( C^c \) are universal constants, not dependent on the choice of \( \mathbb{D} \).

The next lemma shows that these constants converge to the required constant \( C \) in (2.15) as \( \varepsilon \downarrow 0 \).

**Lemma 5.6.** Assume the conditions of Theorem 2.5. Then for any set \( \mathbb{D} = B_\varepsilon (0) \cap \mathbb{R}^d_+ \) with \( \pi (\mathbb{D}) > 0 \),

\[
C_\varepsilon \nearrow C \quad \text{and} \quad C^c \searrow C \quad \text{as} \quad \varepsilon \to 0,
\]

where \( C \) is independent of the choice of \( \mathbb{D} \) and has the representation

\[
C = C (\mathbb{D}) = \int_{\mathbb{D}} r_\alpha (v) E^{\varepsilon}_\delta \left[ \hat{Z}_\varepsilon^\alpha 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \right] \pi (dv).
\]

**Proof.** We claim that

\[
\lim_{\varepsilon \to 0} C_\varepsilon \geq C (\mathbb{D}) \quad \text{and} \quad \lim_{\varepsilon \to 0} C^c \leq C (\mathbb{D}).
\]

Note the these limits necessarily exist, since \( C_\varepsilon, -C^c \) are monotonically increasing (this follows from the monotonicity, in \( \varepsilon \), of \( V_\varepsilon \) and \( V^c \) and the representation (5.12)).

**STEP 1.** We begin by establishing the first inequality in (5.30). Set

\[
H_\varepsilon (v) = r_\alpha (v) E^{\varepsilon}_\delta \left[ \hat{Z}_\varepsilon^\alpha 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \right] \quad \text{and} \quad H (v) = r_\alpha (v) E^{\varepsilon}_\delta \left[ \hat{Z}_\varepsilon^\alpha 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \right].
\]

Then we need to show that

\[
\liminf_{\varepsilon \to 0} \int_{\mathbb{D}} H_\varepsilon (v) \pi_\varepsilon (dv) \geq \int_{\mathbb{D}} H (v) \pi (dv).
\]

We will prove below that: (i) \( H_\varepsilon (v) \uparrow H (v) \) as \( \varepsilon \to 0 \); and (ii) for all \( \varepsilon \geq 0 \), the function \( v \mapsto H_\varepsilon (v) \) is lower semicontinuous.

Assume that (i) and (ii) hold, and fix \( \varepsilon_0 > 0 \). Since \( H_\varepsilon (v) \) is a monotone increasing sequence as \( \varepsilon \downarrow 0 \), we have that

\[
C_\varepsilon \geq \int_{\mathbb{D}} H_\varepsilon (v) \pi_\varepsilon (dv) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0.
\]

As the function \( v \mapsto H_\varepsilon (v) \) is lower semicontinuous and bounded from below by 0, and \( \pi_\varepsilon \Rightarrow \pi \) (cf. (5.22)), the Portmanteau theorem (van der Vaart and Wellner (1996), Theorem 1.3.4 (iv)) yields that

\[
\liminf_{\varepsilon \to 0} C_\varepsilon \geq \liminf_{\varepsilon \to 0} \int_{\mathbb{D}} H_\varepsilon (v) \pi_\varepsilon (dv) \geq \int_{\mathbb{D}} H_\varepsilon (v) \pi (dv).
\]

Now we let \( \varepsilon_0 \to 0 \) and use the monotone convergence \( H_\varepsilon \uparrow H \) to infer, using the monotone convergence theorem,

\[
\liminf_{\varepsilon \to 0} C_\varepsilon \geq \lim_{\varepsilon \to 0} \int_{\mathbb{D}} H_\varepsilon (v) \pi (dv) = \int_{\mathbb{D}} H (v) \pi (dv) = C (\mathbb{D}).
\]

It remains to prove (i) and (ii). In order to obtain (i), observe that \( |V_{n,\varepsilon}| \) increases monotonically to \( |V_\varepsilon| \) as \( \varepsilon \to 0 \). Thus, if the process \( \{V_n\} \) enters \( \mathbb{D} \), then so does \( \{V_{n,\varepsilon}\} \) for all \( \varepsilon > 0 \). Hence, we trivially obtain that \( 1_{\{r (\mathbb{D}) = \infty\}} \geq 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \), where \( r (\mathbb{D}), r_\varepsilon (\mathbb{D}) \) are the first passage times of \( \{V_n\}, \{V_{n,\varepsilon}\} \) into \( \mathbb{D} \), respectively. Conversely, observe that if \( r (\mathbb{D}) = \infty \), then \( V := (V_1, V_2, \ldots) \in (\mathbb{D}^c)^N \), which is open. Now \( V_{\varepsilon} := (V_{1,\varepsilon}, V_{2,\varepsilon}, \ldots) \) converges to \( V \) a.s. in the product topology (as \( \chi_\varepsilon \) is supported on \([-\varepsilon, 0]^d\)). It follows that \( V_\varepsilon \in (\mathbb{D}^c)^N \) for sufficiently small \( \varepsilon \). Consequently, \( 1_{\{r (\mathbb{D}) = \infty\}} \leq \liminf_{\varepsilon \to 0} 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \). Thus we conclude that \( 1_{\{r (\mathbb{D}) = \infty\}} = \lim_{\varepsilon \to 0} 1_{\{r_\varepsilon (\mathbb{D}) = \infty\}} \), and moreover, the convergence is monotone.
i.e. $1_{\{\tau_r(\mathcal{D}) = \infty\}} \uparrow 1_{\{\tau_r(\mathcal{D}) = \infty\}}$ as $\varepsilon \to 0$. Furthermore, as $\hat{Q}_\varepsilon$ increases componentwise to $\hat{Q}$, we deduce from (5.27) and Lemma 3.6 (ii) that $|\hat{Z}_\varepsilon| \uparrow |\hat{Z}|$ as $\varepsilon \to 0$. By Lemma 3.6 (iii), $|\hat{Z}|^\alpha 1_{\{\tau_r(\mathcal{D}) = \infty\}}$ is an integrable upper bound for the family $\{|\hat{Z}_\varepsilon|^\alpha 1_{\{\tau_r(\mathcal{D}) = \infty\}}\}_{\varepsilon > 0}$ and thus we obtain, for all $v \in \mathcal{D}$, the monotone convergence $H_r(v) \uparrow H(v)$.

To obtain (ii), observe that if $v \to \hat{v}$, then $V(v)$ converges to $V(\hat{v})$, where $V := (V_{1,v}, V_{2,v}, \ldots)$ and $V(v)$ denotes the dependence of this quantity on its initial state. Then by repeating the argument given above, we obtain that $\hat{D}$ closed $\implies 1_{\{\tau_{\hat{D},\hat{v}} = \infty\}} \leq \liminf_{v \to \hat{v}} 1_{\{\tau_{\hat{D},\hat{v}} = \infty\}}$, where $\tau(\hat{D}, \cdot)$ again denotes the dependence on the initial state. From the representation (5.27), we obtain that $v \to \hat{Z}_\varepsilon(v)$ is continuous a.s. (the series converges a.s. by Lemma 3.6 (i)). Then we can apply Fatou’s Lemma and use the continuity of $\alpha$ to infer that $H_r(v)$ is lower semicontinuous.

Step 2. To establish the second inequality in (5.30), we proceed as before, now using

\[ (i') : \quad H^\varepsilon(v) := \alpha(v) E^\alpha_{\delta_v} \left[ \hat{Z}^\varepsilon_{\gamma} 1_{\{\tau_r(\mathcal{D}) = \infty\}} \right] \downarrow \alpha(v) E^\alpha_{\delta_v} \left[ \hat{Z}^\varepsilon_{\gamma} 1_{\{\tau_r(\mathcal{D}) = \infty\}} \right] =: H^\varepsilon(v) \]

and (ii') the upper semicontinuity of $H^\varepsilon(v)$, which follows since we consider now the hitting time of an open set. Furthermore, Lemma 3.6 (iii) gives that $\sup_{v \in \mathcal{D}} H^\varepsilon_0(v) \leq B$ for some finite bound $B$. Then we can apply the Portmanteau theorem (van der Vaart and Wellner (1996), Theorem 1.3.4 (v)) to infer that

\[ \limsup_{\varepsilon \to 0} C^\varepsilon \leq \liminf_{\varepsilon \to 0} \int_{\mathcal{D}} H^\varepsilon_0(v) \pi^\varepsilon(dv) = \int_{\mathcal{D}} H^\varepsilon_0(v) \pi(dv) \]

for all $\varepsilon_0 > 0$, and thus, letting $\varepsilon_0 \to 0$ and using (i'),

\[ \limsup_{\varepsilon \to 0} C^\varepsilon \leq \int_{\mathcal{D}} H^\varepsilon(v) \pi(dv) = C(\mathcal{D}) \]

Step 3. Having obtained (5.30), it remains to show that if $\mathcal{D} = B_r(0) \cap \mathbb{R}^d_+$, where $\pi(\mathcal{D}) > 0$, then, in fact,

\[ \lim_{\varepsilon \to 0} C^\varepsilon \geq C(\mathcal{D}). \quad (5.33) \]

[This implies immediately that $\lim_{\varepsilon \to 0} C^\varepsilon = C(\mathcal{D})$ and also that $\lim_{\varepsilon \to 0} C^\varepsilon = C(\mathcal{D})$, since $C^\varepsilon \geq C^\varepsilon$ and $\lim_{\varepsilon \to 0} C^\varepsilon \leq C(\mathcal{D})$ by (5.30).]

To this end, let $\{r_i\}$ be chosen such that $r_i \uparrow r$ as $i \to \infty$ and set $\mathcal{D}_i = B_{r_i}(0) \cap \mathbb{R}^d_+$. If $\{V_n\}$ avoids $\mathcal{D}$, then it also avoids each $\mathcal{D}_i$, so we trivially obtain that $1_{\{\tau_{\mathcal{D}_i} = \infty\}} \geq 1_{\{\tau_{\mathcal{D}} = \infty\}}$. Conversely, $V_n \in \mathcal{D} \implies V_n \in \mathcal{D}_i$ for sufficiently large $i$. Thus $\lim_{i \to \infty} 1_{\{\tau_{\mathcal{D}_i} = \infty\}} = 1_{\{\tau_{\mathcal{D}} = \infty\}}$. Now $\lim_{\varepsilon \to 0} C^\varepsilon$ is a universal constant, independent of the choice of the set $\mathcal{D}$ in (5.30). Consequently, we conclude by (5.30) that

\[ \lim_{\varepsilon \to 0} C^\varepsilon \geq \lim_{i \to \infty} C(\mathcal{D}_i) = \lim_{i \to \infty} \int_{\mathcal{D}_i} \alpha(v) E^\alpha_{\delta_v} \left[ |\hat{Z}|^\alpha 1_{\{\tau_{\mathcal{D}_i} = \infty\}} \right] \pi(dv) \]

\[ = \int_{\mathcal{D}} \alpha(v) E^\alpha_{\delta_v} \left[ |\hat{Z}|^\alpha 1_{\{\tau_{\mathcal{D}} = \infty\}} \right] \pi(dv) = C(\mathcal{D}), \]

as required.

**Proof of Theorem 2.5.** It follows directly from Proposition 5.2 and Lemmas 5.5 and 5.6 that for any $E \in \mathcal{B}(\mathbb{R}^{d-1})$,

\[ \liminf_{u \to \infty} u^\alpha \mathbb{P} \left( |V| > tu, \frac{V}{|V|} \in E \right) \geq \frac{C}{\alpha k \lambda(\alpha)} t^{-\alpha} \limsup_{\varepsilon \to 0} l_\alpha(E_\varepsilon) \quad (5.35) \]

and

\[ \limsup_{u \to \infty} u^\alpha \mathbb{P} \left( |V| > tu, \frac{V}{|V|} \in E \right) \leq \frac{C}{\alpha k \lambda(\alpha)} t^{-\alpha} \liminf_{\varepsilon \to 0} l_\alpha(E_\varepsilon). \quad (5.36) \]

Now if $l_\alpha(\partial E) = 0$, then

\[ \limsup_{\varepsilon \to 0} l_\alpha(E_\varepsilon) = \liminf_{\varepsilon \to 0} l_\alpha(E_\varepsilon) = l_\alpha(E). \]
Hence the bounds coincide and, thus, for all measurable \(E \subset \mathbb{S}^{d-1}_+\) with \(l_\alpha(\partial E) = 0\),

\[
\lim_{u \to \infty} u^\alpha \mathbb{P}\left( |V| > tu, \frac{V}{|V|} \in E \right) = \frac{C}{\alpha k \lambda(\alpha)} t^{-\alpha} l_\alpha(E).
\]

By the Portmanteau Theorem, this implies the weak convergence

\[
u^\alpha \mathbb{P}\left( |V| > tu, \frac{V}{|V|} \in \cdot \right) \Rightarrow \frac{C}{\alpha \lambda(\alpha)} t^{-\alpha} l_\alpha(\cdot) \quad \text{as } u \to \infty,
\]

for all \(t > 0\), which is equivalent to (2.14) by Theorem 2 of Resnick (2004).

\[ \square \]

6 Proof of Theorem 2.9

We now turn to the proof of Theorem 2.9. Note that if \(A\) is a semi-cone, then the event \(\{V_n \in uA, \text{ some } n \leq N\}\) corresponds to the event \(\{\max_{1 \leq n \leq N} |V_n^A| > u\}\), where \(V_n^A := |V_n|/d_A(V_n)\) for \(d_A(x) := \inf\{t > 0 : tx \in A\}\). Thus, as a first step, we study maxima of \(\{|V_n^A|\}\) over cycles emanating from a set \(\mathbb{D}\), where the cycle is terminated upon the return of the process to \(\mathbb{D}\). Consequently, we obtain the asymptotic distribution of \(T_u^A/u^\alpha\), first when \(d_A\) is assumed to be bounded, and then for general sets \(A\). Throughout this section, we assume that the set \(A\) is a semi-cone.

We begin by establishing a preliminary lemma, where we identify the constant appearing in the ruin problem for the Markov random walk \((X_n, S_n^A) : n = 0, 1, \ldots\). For this purpose, define

\[
\mathbb{T}_u^A = \inf \{n \in \mathbb{N} : M_n \cdots M_1 \tilde{V}_0 \in uA\}, \quad \text{where} \quad \tilde{V}_0 = v \in \mathbb{R}^d_+ \setminus \{0\}.
\]

**Lemma 6.1.** Suppose that Hypotheses (H1) and (H2) are satisfied, and assume that \(d_A\) is bounded and continuous on \(\mathbb{S}^{d-1}_+\). Then

\[
\lim_{u \to \infty} u^\alpha \mathbb{P}\left( \mathbb{T}_u^A < \infty \mid V_0 = v \right) = r_\alpha(\tilde{v}) \int_{\mathbb{S}^{d-1}_+} e^{-\alpha \tilde{v}} \frac{d \tilde{v}}{r_\alpha(x)} g^A(dx, ds) := r_\alpha(\tilde{v}) D_A, \tag{6.1}
\]

where \(g^A\) is given as in Theorem 3.9.

**Proof.** Converting to the \(\alpha\)-shifted measure, we obtain for any \(v \in \mathbb{R}^d_+ \setminus \{0\}\) that

\[
u^\alpha \mathbb{P}\left( \mathbb{T}_u^A < \infty \mid V_0 = v \right) = r_\alpha(\tilde{v}) \mathbb{E}_0^\alpha \left( e^{-\alpha(S_{\mathbb{T}_u^A} - \log u)} / r_\alpha(X_{\mathbb{T}_u^A}) \right) 1_{\{\mathbb{T}_u^A < \infty\}}
\]

\[
= r_\alpha(\tilde{v}) \mathbb{E}_0^\alpha \left( e^{-\alpha(S_{\mathbb{T}_u^A} - \log u)} / r_\alpha(X_{\mathbb{T}_u^A}) \right) 1_{\{\mathbb{T}_u^A < \infty\}}, \tag{6.2}
\]

where the last step follows from the definitions of \(S_{\mathbb{T}_u^A}\) and \(r_\alpha^A\). To characterize the limit on the right-hand side, apply Kesten’s renewal theorem (Theorem 3.9) for the bounded continuous function \(g(x, s) = e^{-\alpha s}/r_\alpha^A(x)\). This yields (6.1).

To establish the weak convergence of \(\{T_u^A\}\), the main idea will be to study the excursions of \(\{V_n\}\) over cycles emanating from the set \(\mathbb{D}\). For this purpose, we introduce the random variables

\[
U_i := \max_{\kappa_{i-1} < n \leq \kappa_i} V_n^A, \quad i = 1, 2, \ldots,
\]

where \(V_n^A := |V_n|/d_A(\tilde{V}_n)\), and where \(\kappa_0 = 0\) and \(\kappa_i = \inf\{n > \kappa_{i-1} : V_n \in \mathbb{D}\}\) denote the successive return times to \(\mathbb{D}\). Also set

\[
\mathcal{M}_n^U = \max\{U_1, \ldots, U_n\}, \quad n = 1, 2, \ldots;
\]

\[
\mathcal{M}_n = \max\{V_1^A, \ldots, V_n^A\}, \quad n = 1, 2, \ldots.
\]

Recall that \(\{T_u^A \leq N\} = \{V_n^A > u, \text{ some } n \leq N\}\). Thus, \(\{\mathcal{M}_n^U > u\}\) describes the event that \(T_u^A\) occurs by the random time \(\kappa_n\), while \(\{\mathcal{M}_n > u\}\) describes the event that \(T_u^A\) occurs by the deterministic time \(n\).
Proposition 6.2. Suppose that Hypotheses (H$_1$) and (H$_2$) are satisfied and there exists $m \in \mathbb{N}_+$ such that (H$_3$) holds for the $m$-skeleton $\{V_{mn} : n \in \mathbb{N}\}$. Assume that $\mathbb{D} \subseteq \mathcal{B}(\mathbb{R}^d_+)$ is bounded and $\pi(\mathbb{D}) > 0$, and suppose that the function $d_A$ is bounded and continuous on $\mathbb{S}_+^{d-1}$. Let $\gamma_0$ be an arbitrary probability distribution on $\mathbb{R}^d_+ \setminus \{0\}$. Then

$$\lim_{n \to \infty} \mathbb{P}\left(\mathcal{M}^U_n \leq n^{1/\alpha}u \mid V_0 \sim \gamma_0\right) = \exp\left\{ -K_A \mathbb{E}_{\pi_0}[\tau] u^{-\alpha} \right\},$$

(6.3)

where $K_A = CD_A$ and $C$ is given as in (2.15).

Unless explicitly noted, we assume throughout the rest of this section that $V_0 \sim \gamma_0$, i.e., $\mathbb{P} = \mathbb{P}_{\gamma_0}$.

Proof. Set $u_n = n^{1/\alpha}u$. Then for any $l \in \mathbb{N}_+$,

$$\sum_{i=1}^{l} \mathbb{P}(U_i > u_n) - \sum_{1 \leq i < j \leq l} \mathbb{P}(U_i > u_n, U_j > u_n) \leq \mathbb{P}(\mathcal{M}^U_l > u_n) \leq \sum_{i=1}^{l} \mathbb{P}(U_i > u_n).$$

(6.4)

We begin by calculating $\sum_{i=1}^{n} \mathbb{P}(U_i > u_n)$ as $n \to \infty$ for the sequence $n(n) = [n/k]$ and fixed $k \in \mathbb{N}_+$. By Corollary 4.2 and the Markov property, we have for all $i \in \mathbb{N}_+$ and $v \in \mathbb{D}$ that

$$\lim_{n \to \infty} n u^a \mathbb{P}(U_i > u_n \mid V_{ni-1} = v) = \lim_{n \to \infty} n u^a \mathbb{P}(T^A_u < \tau \mid V_0 = v)$$

$$= r_{\alpha}(v) \mathbb{E}_{\delta_{\alpha}}[\{Z\mid \tau = \infty\}] \int_{\mathbb{R}^d_+} e^{-\alpha s} r_{\alpha}(x) g^A(dv, dx, ds) := H(v) \left( = C(v)D_A \right).$$

(6.5)

[Note that this equation also holds if $i = 0$ and $v \in \mathbb{R}^d_+ \setminus \{0\}$, as $V_{ni}$ need not belong to $\mathbb{D}$.]

Under Hypotheses (H$_3$), $\{V_{mn} : n \geq 0\}$ is a positive aperiodic Harris chain (Lemma 3.4). Hence $\{V_n\}$ is a positive, $m$-periodic Harris chain. Then the hitting chain $\{V_{ni}\}$ is itself a positive $m$-periodic Harris chain as well (Alsmeyer (1991), Theorem 8.3.7), and the invariant measure of this chain is $\pi_{\mathbb{D}}$ (Lemma 3.3). If $\gamma_i$ denotes the law of $V_{ni}$, $i \in \mathbb{N}_+$, then Harris recurrence gives that $\gamma_i \in \gamma_0$. As $n \to \infty$, where $| \cdot |_{TV}$ denotes the total variation distance; see Theorem 13.3.4 in Meyn and Tweedie (1993). Set

$$H_n(v) = n u^a \mathbb{P}(U_i > u_n \mid V_0 = v).$$

By Lemma 4.6 (specifically, (4.21) with $h = 1$), we have that $\sup \{H_n(v) : v \in \mathbb{D}, n \in \mathbb{N}\} \leq B$, for some finite constant $B$. Then $n u^a \mathbb{P}(U_i > u_n) = \int_{\mathbb{D}} H_n(v) \gamma_{i-1}(dv)$, and

$$\left| \frac{1}{n} \sum_{i=1}^{[n/k]} n u^a \mathbb{P}(U_i > u_n) - \int_{\mathbb{D}} H_n(v) \pi_{\mathbb{D}}(dv) \right|$$

$$\leq \left| \int_{\mathbb{D}} H_n(v) \left( \frac{1}{n} \sum_{i=1}^{[n/k]} \gamma_{i-1} - \pi_{\mathbb{D}} \right) (dv) - \int_{\mathbb{D}} (H_n(v) - H(v)) \pi_{\mathbb{D}}(dv) \right|$$

$$\leq B \left| \frac{1}{n} \sum_{i=1}^{[n/k]} \gamma_{i-1} \right|_{TV} + \int_{\mathbb{D}} |H_n(v) - H(v)| \pi_{\mathbb{D}}(dv).$$

(6.6)

The second term tends to zero as $n \to \infty$ by dominated convergence and the fact that $H_n(v) \to H(v)$, by (6.5). Thus the left-hand side of (6.6) tends to zero as $n \to \infty$ and hence, using (6.5),

$$\lim_{n \to \infty} \sum_{i=1}^{[n/k]} u^a \mathbb{P}(U_i > u_n) = \frac{1}{k} \int_{\mathbb{D}} H(v) \pi_{\mathbb{D}}(dv) = \frac{D_A}{k} \int_{\mathbb{D}} C(v) \frac{\pi(dv)}{\pi(\mathbb{D})} = \frac{K_A \mathbb{E}_{\pi_0}[\tau]}{k},$$

(6.7)

since $C = \int_{\mathbb{D}} C(v) \pi(dv)$ and $\mathbb{E}_{\pi_0}[\tau] = (\pi(\mathbb{D}))^{-1}$, by Lemma 3.3. Using this equation in (6.4), we deduce that for any $k \in \mathbb{N}_+$,

$$\limsup_{n \to \infty} \mathbb{P}\left(\mathcal{M}^U_{[n/k]} > u_n\right) \leq \frac{K_A \mathbb{E}_{\pi_0}[\tau]}{k} u^{-\alpha}.$$

(6.8)
Note that the right-hand side is independent of \( V_0 \in \mathbb{R}^d \setminus \{0\} \). Since this quantity is asymptotically independent of the initial state, the same calculation yields the asymptotic behavior of the maximum over any block of length \( n/k \); more precisely, for \( \lim_{n \to \infty} \mathbb{P} \left( U_{\lfloor jn/k \rfloor + 1}, \ldots, U_{\lfloor (j+1)n/k \rfloor} > u_n \mid \mathcal{F}_{\lfloor jn/k \rfloor} \right) \), \( j = 0, \ldots, k - 1 \). Hence we conclude from (6.8) that

\[
\limsup_{n \to \infty} \mathbb{P} (\mathcal{M}_n^U \leq u_n) \leq \left( 1 - \frac{K_A \mathbb{E}_{\pi_0}[\tau] u^{-\alpha}}{k} \right)^k \to \exp \left\{ -K_A \mathbb{E}_{\pi_0}[\tau] u^{-\alpha} \right\} \quad \text{as } k \to \infty. \tag{6.9}
\]

Moreover, using once again the uniform upper bound (in the initial state) provided by Lemma 4.6, we obtain that for any positive integer \( k \),

\[
\limsup_{n \to \infty} \sum_{1 \leq i < j \leq \lfloor n/k \rfloor} \mathbb{P} (U_i > u_n, U_j > u_n) = o \left( \frac{1}{k} \right) \quad \text{as } n \to \infty. \tag{6.10}
\]

Finally, using (6.7) and (6.10) in (6.4), we conclude that

\[
\liminf_{n \to \infty} \mathbb{P} (\mathcal{M}_n^U \leq u_n) \geq \exp \left\{ -K_A \mathbb{E}_{\pi_0}[\tau] u^{-\alpha} \right\}. \tag*{\Box}
\]

**Lemma 6.3.** Suppose that Hypotheses (H1)–(H3) are satisfied and the function \( d_A \) is bounded and continuous. Then for any \( \Delta > 0 \), there exists a constant \( \delta > 0 \) such that

\[
\limsup_{n \to \infty} \mathbb{P} \left( \max_{|m-n| \leq n\delta} |\mathcal{M}_m - \mathcal{M}_n| > n^{1/\alpha} \Delta \right) \leq \Delta. \tag{6.11}
\]

**Proof.** Let \( k \in \mathbb{N}_+ \). Then

\[
\mathcal{M}_{n+k} = \max \left\{ \mathcal{M}_n, V_{n+1}^A, \ldots, V_{n+k}^A \right\} \Rightarrow \mathcal{M}_n \leq \mathcal{M}_{n+k} \leq \mathcal{M}_n + \max \left\{ V_{n+1}^A, \ldots, V_{n+k}^A \right\}. \]

Since \( V_n^A := |V_n|/d_A(\bar{V}_n) \), it follows that

\[
\max_{|m-n| \leq n\delta} |\mathcal{M}_m - \mathcal{M}_n| \leq b \max \left\{ |V_{n-n\delta}^A|, \ldots, |V_{n+n\delta}^A| \right\}, \tag{6.12}
\]

where \( b = \max \left\{ \left( d_A(x) \right)^{-1} : x \in S_+^{d-1} \right\} < \infty \). We now determine the maximum on the right-hand side conditioned on \( V_{n-n\delta} = v \). Equivalently, we study \( \mathcal{M}_{m_n} \) conditioned on \( V_0 = v \), where \( m_n = [n+n\delta] - ([n-n\delta] + 1) \leq 2n\delta \).

Let \( D \subset \mathbb{R}_+^d \setminus \{0\} \) be chosen such that \( \pi(D^c) \leq \Delta/2 \), and let \( v \in D \). Since \( \mathcal{M}_n \leq \mathcal{M}_n^U \) and \( m_n \leq 2n\delta \), we obtain from Proposition 6.2 with \( A = \{ x \in \mathbb{R}_+^d : |x| > 1 \} \) that

\[
\limsup_{n \to \infty} \mathbb{P} \left( \mathcal{M}_{m_n} > n^{1/\alpha} \Delta \mid V_0 = v \right) \leq 1 - \exp \left\{ -K_A \mathbb{E}_{\pi_0}[\tau] \cdot 2\delta \Delta^{-\alpha} \right\}
\]

\[
= 2K_A \mathbb{E}_{\pi_0}[\tau] \Delta^{-\alpha} t, \quad \text{where } t \in (0, \delta), \tag{6.13}
\]

and the right-hand side is \( \leq \Delta/2 \) when \( \delta \) is chosen sufficiently small. Note that (6.13) holds for all \( v \in D \). Finally, let \( \gamma_n \) denote the distribution function of \( V_{n-n\delta} \). By the positive Harris recurrence of \( \{V_n\} \) (Lemma 3.4), we have that \( |\gamma_n - \pi|TV \to 0 \) as \( n \to \infty \). Then, using Fatou’s lemma, we deduce that

\[
\limsup_{n \to \infty} \mathbb{P} \left( \max_{|m-n| \leq n\delta} |\mathcal{M}_m - \mathcal{M}_n| > n^{1/\alpha} \Delta \right) \leq \limsup_{n \to \infty} \left( \gamma_n(D^c) + |\gamma_n - \pi|TV + \int_D \mathbb{P} \left( \mathcal{M}_{m_n} > n^{1/\alpha} \Delta \mid V_0 = v \right) \pi(dv) \right)
\]

\[
\leq \pi(D^c) + \frac{\Delta}{2} \pi(D) \leq \Delta. \tag*{\Box}
\]
Proof of Theorem 2.9. Assuming that $d_A$ is bounded, the first assertion follows from Corollary 4.2 and from the uniformity provided by Lemma 4.6. To remove the assumption that $d_A$ is bounded, see Step 4 below.

To establish the remaining assertion, we proceed in four steps.

STEP 1. First assume that $d_A$ is bounded and continuous and that $(H_3)$ is satisfied. We claim that
\begin{equation}
\lim_{n \to \infty} \mathbb{P}(\mathcal{M}_n \leq n^{1/\alpha} u) = e^{-K_A u^{-\alpha}}; \tag{6.14}
\end{equation}
that is, we can transfer the result for maxima over cycles (Proposition 6.2) to the process of running maxima, namely to $\mathcal{M}_n$.

To establish an upper bound for $\limsup_{n \to \infty} \mathbb{P}(\mathcal{M}_n \leq n^{1/\alpha} u)$ observe that, by definition, $\mathcal{M}^U_{N_\mathbb{D}(n)}$ corresponds to the value of the process $\{\mathcal{M}_j\}$ evaluated at the time of its last visit to $\mathbb{D}$ within the time interval $[0, n]$. Thus
\begin{equation}
\mathbb{P}(\mathcal{M}_n > n^{1/\alpha} u) \geq \mathbb{P}(\mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} u). \tag{6.15}
\end{equation}
To replace the random time $N_\mathbb{D}(n)$ by a fixed time, observe by Lemma 3.3 that for all $\delta > 0$,
\begin{equation}
\mathbb{P}\left(\left| N_{\mathbb{D}}(n) - \pi(\mathbb{D}) \right| \geq \delta \right) \to 0 \text{ as } n \to \infty. \tag{6.16}
\end{equation}
Set $t_n = n (\pi(\mathbb{D}) - \delta)$ and $\Omega_n = \{|(N_{\mathbb{D}}(n)/n) - \pi(\mathbb{D})| < \delta\}$, and note that $N_{\mathbb{D}}(n) \geq |t_n|$ on $\Omega_n$. Then
\begin{equation}
\mathbb{P}(\mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} u) \geq \mathbb{P}(\mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} u; \Omega_n) \geq \mathbb{P}(\mathcal{M}^U_{t_n} > n^{1/\alpha} u) - \mathbb{P}(\Omega_n^c). \tag{6.17}
\end{equation}
Then combining (6.15), (6.16), and (6.17) and applying Proposition 6.2, we conclude that for all $\delta > 0$,
\begin{equation}
\liminf_{n \to \infty} \mathbb{P}(\mathcal{M}_n > n^{1/\alpha} u) \geq \lim_{n \to \infty} \mathbb{P}(\mathcal{M}^U_{t_n} > n^{1/\alpha} u) = 1 - \exp\left\{-K_A\mathbb{E}_{\tau_0}[\tau] (\pi(\mathbb{D}) - \delta) u^{-\alpha}\right\}
\end{equation}
and hence, letting $\delta \downarrow 0$ and recalling that $\pi(\mathbb{D}) = (\mathbb{E}_{\pi_\mathbb{D}}[\tau])^{-1}$ (Lemma 3.3), we obtain that
\begin{equation}
\limsup_{n \to \infty} \mathbb{P}(\mathcal{M}_n \leq n^{1/\alpha} u) \leq \exp\{-K_A u^{-\alpha}\}. \tag{6.18}
\end{equation}

To establish the corresponding lower bound for $\mathbb{P}(\mathcal{M}_n \leq n^{1/\alpha} u)$, observe that for any $\Delta > 0$,
\begin{equation}
\mathbb{P}(\mathcal{M}_n > n^{1/\alpha} u) \leq \mathbb{P}(\mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} (u - \Delta)) + \mathbb{P}(\mathcal{M}_n - \mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} \Delta). \tag{6.19}
\end{equation}
Reasoning as before, we obtain that the first term on the right-hand side of (6.19) satisfies
\begin{equation}
\limsup_{n \to \infty} \mathbb{P}(\mathcal{M}^U_{N_\mathbb{D}(n)} > n^{1/\alpha} (u - \Delta)) \leq 1 - \exp\{-K_A(u - \Delta)^{-\alpha}\}. \tag{6.20}
\end{equation}

To quantify the second term on the right-hand side of (6.19), first note that $\mathcal{M}^U_{N_\mathbb{D}(n)}$ corresponds to the value of the process $\{\mathcal{M}_j\}$ evaluated at the time of its last visit to $\mathbb{D}$ in the interval $[0, n]$. Since $\kappa_i$ denotes the time of the $i^{th}$ visit to $\mathbb{D}$, this gives $\mathcal{M}^U_{N_\mathbb{D}(n)} = \mathcal{M}_{\kappa_{N_\mathbb{D}(n)}}$. Moreover, for any $\delta > 0$, it follows from Lemma 3.3 that
\begin{equation}
\mathbb{P}\left(\left| \frac{\kappa_{N_\mathbb{D}(n)} - 1}{n} \right| \geq \delta \right) \to 0 \text{ as } n \to \infty.
\end{equation}
Set $\tilde{\Omega}_n = \{|(\kappa_{N_\mathbb{D}(n)}/n - 1| < \delta\}$. Then
\begin{equation}
\mathbb{P}\left(\left| \mathcal{M}_n - \mathcal{M}^U_{N_\mathbb{D}(n)} \right| > n^{1/\alpha} \Delta \right) \leq \mathbb{P}\left(\left| \mathcal{M}_n - \mathcal{M}^U_{N_\mathbb{D}(n)} \right| > n^{1/\alpha} \Delta; \tilde{\Omega}_n \right) + \mathbb{P}(\tilde{\Omega}_n^c)
\end{equation}
\begin{equation}
\leq \mathbb{P}\left(\max_{|m-n|<n\delta} |\mathcal{M}_m - \mathcal{M}_n| > n^{1/\alpha} \Delta \right) + o(1)
\end{equation}

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as $n \to \infty$. Hence by Lemma 6.3,
\[
\limsupe \lim_{n \to \infty} P \left( | \mathcal{M}_n - \mathcal{M}^{\hat{L}}_{N_0}(n) | > n^{1/\alpha} \Delta \right) \leq \Delta.
\] (6.21)

Finally, substituting (6.20) and (6.21) into (6.19) and letting $\Delta \to 0$, we obtain that
\[
\limsupe \lim_{n \to \infty} P \left( \mathcal{M}_n \leq n^{1/\alpha} u \right) \geq \exp \{ -K_A u^{-\alpha} \}.
\]
Together with (6.18), this implies the assertion.

**Step 2.** Next we remove the additional assumption $(H_3)$, but still assume that the function $d_A$ is bounded and continuous.

To remove $(H_3)$, we employ the smoothing argument introduced in Section 5. Let $\{ (\tilde{M}_n, \tilde{Q}_n : n = 0, 1, \ldots \}$ be defined as in (5.18). Then, since we have assumed that $k = 1$ in (9), it follows that $(\tilde{M}_n, \tilde{Q}_n) = (M_n, Q_n)$ for all $n \in \mathbb{N}_+$. This gives that $V_{n, \varepsilon} \leq V_n \leq V_n^\varepsilon$ for all $n \in \mathbb{N}_+$.

By repeating the computation leading to (5.25), we obtain that
\[
\left| \tilde{V}_{n, \varepsilon} - \tilde{V}_n \right| \leq \frac{2\varepsilon}{s} \quad \text{and} \quad \left| V_{n, \varepsilon} - V_n \right| \leq \frac{2\varepsilon}{s} \quad \text{for all} \ n \in \mathbb{N}_+.
\]
Since the function $d_A$ is assumed to be continuous on the compact set $S_{+}^{d-1}$, it is, in fact, equicontinuous. Hence there is a sequence $\delta(\varepsilon)$, tending to zero as $\varepsilon \to 0$, such that
\[
\left| d_A(\tilde{V}_{n, \varepsilon}) - d_A(\tilde{V}_n) \right| \leq \delta(\varepsilon) \quad \text{and} \quad \left| d_A(\tilde{V}_{n, \varepsilon}^\varepsilon) - d_A(\tilde{V}_n) \right| \leq \delta(\varepsilon) \quad \text{for all} \ n \in \mathbb{N}_+.
\] (6.22)

Since $A \subset \{ v : |v| > 1 \} \implies d_A > 1$, it follows that for all $n \in \mathbb{N}_+$,
\[
\left| V_n \right| \leq u \cdot d_A(\tilde{V}_n) \implies \left| V_{n, \varepsilon} \right| \leq u \cdot d_A(\tilde{V}_n) \leq u \cdot d_A(\tilde{V}_{n, \varepsilon}) + u \cdot \delta(\varepsilon) \leq u \cdot d_A(\tilde{V}_{n, \varepsilon})(1 + \delta(\varepsilon)).
\]
Consequently, $P(\mathcal{M}_n \leq u) \leq P(\mathcal{M}_n \leq u(1 + \delta(\varepsilon)))$. Similarly, for all $n \in \mathbb{N}_+$,
\[
\left| V_{n, \varepsilon} \right| \leq u \cdot d_A(\tilde{V}_{n, \varepsilon})(1 - \delta(\varepsilon)) \implies \left| V_n \right| \leq u \cdot d_A(\tilde{V}_{n, \varepsilon}) - u \cdot \delta(\varepsilon) \leq u \cdot d_A(\tilde{V}_{n, \varepsilon}),
\]
and we obtain that $P(\mathcal{M}_n \leq u) \geq P(\mathcal{M}_n \leq u(1 - \delta(\varepsilon)))$. Using these upper and lower bounds together with Step 1, we conclude that
\[
\exp \{ -K_\varepsilon(u - \delta(\varepsilon))^{-\alpha} \} \leq \limsupe \lim_{n \to \infty} P \left( \mathcal{M}_n \leq n^{1/\alpha} u \right)
\leq \limsupe \lim_{n \to \infty} P \left( \mathcal{M}_n \leq n^{1/\alpha} u \right) \leq \exp \{ -K_\varepsilon(u + \delta(\varepsilon))^{-\alpha} \},
\] (6.23)
for constants $K_\varepsilon := C_\varepsilon D_A$ and $K^{\varepsilon} := C^{\varepsilon} D_A$, where $C_\varepsilon$ and $C^{\varepsilon}$ are given as in Section 5. Now by Lemma 5.6, $\sup_{\varepsilon \to 0} C_\varepsilon \leq C$ and $\liminf_{\varepsilon \to 0} C^{\varepsilon} \geq C$. Thus, letting $\varepsilon \downarrow 0$, we obtain (6.14).

**Step 3.** We now relate the behavior of the maxima to the behavior of the first passage times. Recall that $V_n \in uA \iff |V_n| > u \cdot d_A(V_n) \iff V_n^A > u$. Hence for all $n \in \mathbb{N}_+$ and all $u > 0$,
\[
P(T_n^A \leq u) = P(V_i^A > u, \text{ some } 1 \leq i \leq n) = P(\mathcal{M}_n > u).
\] (6.24)
Then by (6.14) and (6.24),
\[
\lim_{n \to \infty} P \left( T_n^{1/\alpha} u \leq n \right) = 1 - e^{-K_A u^{-\alpha}}.
\]
Finally, setting $u = n^{1/\alpha} v$ and $z = v^{-\alpha}$ yields
\[
\lim_{v \to \infty} P \left( \frac{T_n^A}{u^{1/\alpha}} \leq z \right) = 1 - e^{-K_A z}, \quad z \geq 0.
\] (6.25)

**Step 4.** Finally suppose that $\mathcal{P}_A := \{ x \in S_{+}^{d-1} : d_A(x) < \infty \} \neq S_{+}^{d-1}$. For any $L \geq 1$, set
\[
\mathcal{X}_L = \{ v \in \mathbb{R}_+^d : |v| \geq L \} \quad \text{and} \quad A_L = A \cup \mathcal{X}_L.
\]
First observe that $d_{\mathcal{X}_L}(x) := \inf\{t : tx \in \mathcal{X}_L\} = L$, for all $x \in S_{n,1}^{d-1}$. Hence, letting $r^L_{\alpha}$ be defined as in (2.21), we have that $r^L_{\alpha}(x) = L^\alpha r_\alpha(x) \uparrow \infty$ as $L \to \infty$, uniformly in $x$, since $r_\alpha$ is uniformly bounded from below by a positive constant, by Lemma 2.2. Now in general, the constant $K_A$ is proportional to $D_A$, where the latter constant was characterized in Lemma 6.1. Using this characterization, we see that $r^L_{\alpha}(x) \uparrow \infty \forall x \implies D^L_{\alpha} \downarrow 0$ as $L \to \infty$. Consequently,

$$\Delta(L) := \lim_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) \geq 0 \quad \text{as} \quad L \to \infty. \quad (6.26)$$

Since

$$\left| \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) - \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) \right| \leq \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right),$$

we conclude that for all $z \geq 0$,

$$-\Delta(L) + \lim_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) \leq \lim_{L \to \infty} \inf_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) \leq \lim_{L \to \infty} \sup_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) \leq \Delta(L) + \lim_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right). \quad (6.27)$$

Thus, by (6.26) and Step 3,

$$\lim_{u \to \infty} \mathbb{P}\left(\frac{T^L_{\alpha}}{u\alpha} \leq z\right) = 1 - \lim_{L \to \infty} \exp\{ -CD_{AL}z \} := 1 - \exp\{ -CD_{AZ}z \}. \quad (6.28)$$

Observe that $D_A := \lim_{L \to \infty} D_{AL}$ exists, since $D_{AL} = u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = \bar{v}\right)$ is a decreasing sequence; that is to say, the hitting probability of a decreasing sequence of sets.

It remains to identify $D_A$ as the ruin constant in this case. Arguing as before, we have for all $u > 0$ that

$$\left| u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = \bar{v}\right) - u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = v\right) \right| \leq u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = v\right) \geq 0 \quad \text{as} \quad L \to \infty.$$

Thus, we deduce by another sandwich argument that

$$\lim_{u \to \infty} u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = v\right) = \lim_{L \to \infty} \left( \lim_{u \to \infty} u^\alpha \mathbb{P}\left(\bar{\xi}^L_{u \alpha} < \infty | V_0 = \bar{v}\right) \right) = r_\alpha(\bar{v}) \lim_{L \to \infty} D_{AL} = r_\alpha(\bar{v}) D_A,$$

which gives the required identification of $D_A$ as the constant in the ruin problem for the Markov random walk; cf. Lemma 6.1.

To conclude the proof, observe that the same reasoning yields (2.25) for unbounded functions $d_A$; namely, one can again introduce the family $A_L = A \cup \mathcal{X}_L$ for $L > 1$, and argue that the hitting probability of the set $\mathcal{X}_L$—now prior to the return time $\tau$—becomes asymptotically negligible as $L \to \infty$. The argument is entirely identical, so we omit the details.

7 Determining the path of large exceedance

We conclude by studying the path of large exceedance conditioned on $\{T^L_{\alpha} \leq \tau\}$. In particular, we provide the proofs of Theorem 2.14 and a stronger version of Theorem 2.13, where we also allow for paths of infinite length.

To study paths of infinite length in the context of Theorem 2.13, first introduce the normalized process

$$\tilde{S}_n = S_n - n \log a, \quad n = 0, 1, \ldots,$$

where $a \geq 0$. A natural choice is $\log a = \Lambda'/a = \tilde{E}^{\alpha}[S_1]$, in which case $n^{-1} \tilde{S}_n \to 0$ a.s. as $n \to \infty$, by Lemma 2.3. Similarly, we introduce a normalization for $\{V_n\}$, describing the behavior after this process has exceeded an initial barrier $u_\epsilon$, where $u_\epsilon = \epsilon(u)$ and $u_\epsilon \uparrow \infty$ as $u \to \infty$. Conditioned on
our objective is to characterize \( \{ V_n \} \) over the time interval \( [T_u, T_u^A] \), and to show that this process resembles the process \( \{ e^{S_n} X_n \} \) under \( \mathbb{P}^\alpha \).

Let
\[
\gamma_u = u - \varepsilon_u, \quad \text{and set} \quad I_u = T_{\varepsilon_u} \quad \text{and} \quad J_u = T_u^A \quad (7.1)
\]

Now by the nonlinear renewal theory in Subsection 3.3 (cf. Eq. (6.33) in Lemma 3.15), \( \log |V_{u+n}| - \log |V_{u}| \) grows under \( \mathbb{P}^\alpha \) at roughly the rate \( n \Lambda'(\alpha) \) as \( n \to \infty \), where \( \Lambda'(\alpha) \) represents the mean of the Markov random walk \( \{ (X_n, \tau_n) \} \). Thus, to describe the post-\( I_u \) behavior of \( \{ V_n \} \), it is natural to consider the normalized process
\[
\bar{V}^{(u)}_{I_u+n} := \frac{1}{\alpha^n} V_{I_u+n} / |V_{I_u}|, \quad n = 0, 1, \ldots,
\]
where \( \log a = \Lambda'(\alpha) \) and and \( \bar{V}^{(u)}_{k} := 0 \) for all \( k < I_u \). In practice, when studying the behavior of \( \{ V_n \} \) over a path of infinite length, we generally need to consider other choices of \( a \), where \( \log a > \Lambda'(\alpha) \), and thus \( a \) should be viewed here as an arbitrary parameter subject to the constraint that \( \log a \geq 0 \).

Finally, given a measurable function \( g : (\mathbb{R}^d)^N \to \mathbb{R} \) and \( m \in \mathbb{N}_+ \), define \( \mathcal{P}_{[m]} g : (\mathbb{R}^d)^m \to \mathbb{R} \) by setting \( \mathcal{P}_{[m]} g(x_1, \ldots, x_m) = g(x_1, \ldots, x_m, 0, 0, \ldots) \). Thus, \( \mathcal{P}_{[m]} g \) is determined by \( g \) under the projection of \( (x_1, x_2, \ldots) \) onto \( (x_1, \ldots, x_m) \). Also, extending the standard finite-dimensional definition, we say that \( g : (\mathbb{R}^d)^N \to \mathbb{R} \) is \( \theta \)-Hölder continuous if for all sequences \( \{ u_i \}_{i \in \mathbb{N}_+} \), \( \{ v_i \}_{i \in \mathbb{N}_+} \in (\mathbb{R}^d)^N \),
\[
\left| g(x_1, x_2, \ldots) - g(y_1, y_2, \ldots) \right| \leq \sum_{i \geq 1} |x_i - y_i|^\theta.
\]

In the next theorem, note that \( \mathcal{P}_{[k]} g = \mathcal{P}_{[m]} g \) for all \( k \geq m \iff g(x_1, x_2, \ldots) = g(x_1, \ldots, x_m, 0, 0, \ldots) = g^*(x_1, \ldots, x_m) \). Recall that \( \mathcal{D} \) denotes the intersection of the domain of \( \Lambda \) with \( \mathbb{R}_+ \).

**Theorem 7.1.** Suppose that Hypotheses (H1) and (H2) are satisfied, and assume that the function \( d_A \) is finite and continuous on \( S_{d-1}^+ \). Let \( \{ \varepsilon_u : u \in \mathbb{R}_+ \} \) be any sequence such that \( \varepsilon_u = o(u) \) and \( \varepsilon_u \to \infty \) as \( u \to \infty \). Let \( I_u \) and \( J_u \) be defined as in (7.1), and set \( g_u = \mathcal{P}_{[J_u-I_u]} g \), where \( g : \mathbb{R}^N \to \mathbb{R} \) is a bounded measurable function. Assume that either:

(i) \( \mathcal{P}_{[k]} g = \mathcal{P}_{[m]} g \) for all \( k \geq m \), some \( m \in \mathbb{N}_+ \), and that \( g \) is \( \theta \)-Hölder continuous for some \( \theta \leq \min\{1, \alpha\} \); or

(ii) \( g \) is \( \theta \)-Hölder continuous for some \( 0 < \theta \leq 1 \) satisfying \( \alpha + \theta \in \mathcal{D} \), where \( \mathbb{E}[|M|^\alpha |Q|^\theta] < \infty \) and \( \log a > \Lambda(\alpha + \theta)/\theta \).

Then for all \( v \in \mathbb{R}^d_+ \),
\[
\lim_{u \to \infty} \mathbb{E} \left[ g_u \left( \bar{V}^{(u)}_{I_u}, \ldots, \bar{V}^{(u)}_{J_u} \right) \mathbf{1}_{T_u^A < \tau} \left| V_0 = v, T_u^A < \tau \right. \right.
\]
\[
= \int_{S_{d-1}^+ \times \mathbb{R}_+} \mathbb{E}_x^\alpha \left[ g(X_0, e^{S_1} X_1, e^{S_2} X_2, \ldots) \right] \rho(dx, ds). \quad (7.2)
\]

Note by definition that \( |\bar{V}^{(u)}_{I_u}| = 1 \), and consequently the initial value on the right-hand side is \( X_0 \in S_{d-1}^+ \), not \( e^* X_0 \) as would appear if, instead, we had normalized \( V_{I_u} \) by dividing by \( \varepsilon_u \). Further, since \( \Lambda(A) \) is a convex function, we have that \( \Lambda(\alpha + \theta) \geq \Lambda'(\alpha) \theta \) for \( \Lambda(\alpha) = 0 \). Thus, in (ii), we require \( \log a > \Lambda'(\alpha) \).

**Proof.** To establish the result, it suffices to prove that
\[
\lim_{u \to \infty} u^\alpha \mathbb{E} \left[ g_u \left( \bar{V}^{(u)}_{I_u}, \ldots, \bar{V}^{(u)}_{J_u} \right) \mathbf{1}_{T_u^A < \tau} \left| V_0 = v \right. \right. \]
\[
= r_n(\bar{v}) \mathbb{E}_x^\alpha \left[ |Z|^\alpha \mathbf{1}(\tau = \infty) \right]
\]
\[
\times \int_{S_{d-1}^+ \times \mathbb{R}_+} \mathbb{E}_x^\alpha \left[ g(X_0, e^{S_1} X_1, e^{S_2} X_2, \ldots) \right] \rho(dx, ds) \left( \int_{S_{d-1}^+ \times \mathbb{R}_+} e^{-a} \right)^\alpha \rho^\Lambda(dx, ds). \quad (7.3)
\]
Once (7.3) is established, then the assertion follows by using (7.3) twice (once with \( g = 1 \)), observing that
\[
\mathbb{E}\left[ g_u \left( \bar{V}_{I_n}^{(u)}, \ldots, \bar{V}_{J_n}^{(u)} \right) \mid V_0 = v, T_u^A < \tau \right] = \frac{\mathcal{A}_1}{\mathcal{A}_2}, \tag{7.4}
\]
where
\[
\mathcal{A}_1 = u^\alpha \mathbb{E}\left[ g_u \left( \bar{V}_{I_n}^{(u)}, \ldots, \bar{V}_{J_n}^{(u)} \right) 1_{\{T_u^A < \tau\}} \mid V_0 = v \right] \quad \text{and} \quad \mathcal{A}_2 = u^\alpha \mathbb{E}\left[ 1_{\{T_u^A < \tau\}} \mid V_0 = v \right].
\]

To verify (7.3), we proceed as in the proof of Proposition 4.1, converting to the \( \alpha \)-shifted measure to obtain that
\[
u^\alpha \mathbb{E}\left[ g_u \left( \bar{V}_{I_n}^{(u)}, \ldots, \bar{V}_{J_n}^{(u)} \right) 1_{\{T_u^A < \tau\}} \mid V_0 = v \right] = \nu^\alpha r_\alpha(\bar{v}) \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ e^{-\alpha S_{T_u^A}} g_u \left( \bar{V}_{I_n}^{(u)}, \ldots, \bar{V}_{J_n}^{(u)} \right) \mid T_u^A < \tau \right]
\]
\[
= r_\alpha(\bar{v}) \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ |Z_{T_u^A}|^\alpha \Theta_u 1_{\{T_u^A < \tau\}} \right], \tag{7.5}
\]
where \( Z_n := |V_n|/e^{S_n}, \ n = 0, 1, \ldots, \) and
\[
\Theta_u := \frac{1}{r_\alpha'(X_{T_u^A})} \left( \frac{|V_{T_u^A}|}{u} \right)^{-\alpha} \left( \frac{d_A(X_{T_u^A})}{d_A(\bar{V}_{T_u^A})} \right)^\alpha g_u \left( \bar{V}_{I_n}^{(u)}, \ldots, \bar{V}_{J_n}^{(u)} \right). \tag{7.6}
\]

[The term \( \left( d_A(X_{T_u^A})/d_A(\bar{V}_{T_u^A}) \right)^\alpha \) arises when we replace \( |V_{T_u^A}|^{-\alpha}/r_\alpha(X_{T_u^A}) \) with \( |V_{T_u^A}|^{-\alpha}/r_A(X_{T_u^A}) \).]

The right-hand side of (7.5) can be written as the difference of two terms, namely
\[
r_\alpha(\bar{v}) \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ \left( |Z_{T_u^A}|^\alpha 1_{\{T_u^A < \tau\}} - |Z_n|^\alpha 1_{\{n \leq T_u^A\} 1_{\{n \leq \tau\}}} \right) \Theta_u \right]
\]
\[
+ r_\alpha(\bar{v}) \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ |Z_n|^\alpha 1_{\{n \leq T_u^A\} 1_{\{n \leq \tau\}}} \mathbb{E}^\alpha \left[ \Theta_u \mid \mathcal{F}_n \right] \right]. \tag{7.7}
\]

As in the proof of Proposition 4.1, we may then apply Lemma 4.4 (i) and use the uniform boundedness of \( \{\Theta_u\} \) to conclude that the first term in (7.7) tends to zero as \( u \to \infty \) and then \( n \to \infty \).

Thus, it suffices to analyze the second term in (7.7). Indeed, the proof of the theorem will be complete once we have established the following.

**Lemma 7.2.** Under the conditions of Theorem 7.1,
\[
\lim_{n \to \infty} \lim_{u \to \infty} \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ |Z_n|^\alpha 1_{\{n \leq T_u^A\} 1_{\{n \leq \tau\}}} \mathbb{E}^\alpha \left[ \Theta_u \mid \mathcal{F}_n \right] \right]
\]
\[
= H r_\alpha(\bar{v}) \mathbb{E}_{\mathcal{G}_u}^\alpha \left[ |Z|^\alpha 1_{\{\tau = \infty\}} \right] \int_{\mathbb{S}_{+}^{d-1} \times \mathbb{R}^+} \mathbb{E}_{\mathcal{x}}^\alpha \left[ g(X_0, e^{S_1} X_1, e^{S_2} X_2, \ldots) \right] \vartheta(dx, ds), \tag{7.8}
\]
where \( H := \int_{\mathbb{S}_{+}^{d-1} \times \mathbb{R}^+} (e^{-\alpha S_1}/r_\alpha(x)) \vartheta^A(dx, ds) \).

**Proof of Lemma 7.2.** Step 1. We begin by analyzing \( \mathbb{E}^\alpha[\Theta_u \mid \mathcal{F}_n] \), showing that there is asymptotic independence between \( \{V_n : n \leq J_u\} \) and \( V_{T_u^A}/u \). Set
\[
h(u) = \mathbb{E}^\alpha \left[ \frac{1}{r_\alpha'(V_{T_u^A})} \left( \frac{|V_{T_u^A}|}{u} \right)^{-\alpha} \left( \frac{d_A(X_{T_u^A})}{d_A(\bar{V}_{T_u^A})} \right)^\alpha \frac{r_\alpha'(\bar{V}_{T_u^A})}{r_\alpha'(X_{T_u^A})} \mid \mathcal{F}_{J_u} \right]. \tag{7.9}
\]
Recall that \( J_u = T_u - \varepsilon_u \), where \( u - \varepsilon_u \to \infty \). Now decompose \( h \) into two parts, namely

\[
h(u) = h(u) \mathbb{1}_{\{|V_{j_u}^A| \leq u - \frac{x}{2}\}} + h(u) \mathbb{1}_{\{|V_{j_u}^A| > u - \frac{x}{2}\}},
\]

and observe that the second term tends to zero in \( \mathbb{P}^\alpha \)-probability, since the sequence \((u - \varepsilon_u)^{-1}|V_{j_u}^A|\) is tight, by Lemma 3.13.

To analyze the first term, we employ nonlinear renewal theory. Since \( d_A^\alpha \) and \( r_A^\alpha \) are continuous functions and bounded from below, Lemma 3.12 implies that \((d_A(X_{T_u^A})/d_A(\tilde{V}_{T_u^A}))^\alpha\) and \(r_A^\alpha(\tilde{V}_{T_u^A})/r_A^\alpha(X_{T_u^A})\) tend to one in \( \mathbb{P}^\alpha \)-probability. Moreover, by the continuous mapping theorem and Theorem 3.10, \((r_A^\alpha(\tilde{V}_{T_u^A}))^{-1}\exp(-\alpha(\log V_{j_u}^A - \log u))\) converges in law, independent of the initial distribution. Then by the Markov property and Slutsky’s theorem,

\[
h(u) \mathbb{1}_{\{|V_{j_u}^A| \leq u - \frac{x}{2}\}} = \mathbb{E}^X_{j_u, V_{j_u}} \left[ \exp \left\{ -\alpha \left( \log V_{j_u}^A - \log u \right) \right\} \frac{d_A(X_{T_u^A})}{d_A(\tilde{V}_{T_u^A})} \frac{\alpha}{r_A^\alpha(\tilde{V}_{T_u^A})} \right] \mathbb{1}_{\{|V_{j_u}^A| \leq u - \frac{x}{2}\}}
\]

\[
\rightarrow \int_{[0,\infty) \times \mathbb{S}^{d-1}} \frac{e^{-\alpha s}}{r_A^\alpha(x)} g^A(dx, ds) := H \quad \text{in} \quad \mathbb{P}^\alpha \text{-probability as} \ u \to \infty. \quad (7.10)
\]

Thus we obtain the convergence \( h(u) \to H \) in \( \mathbb{P}^\alpha \)-probability. Next observe that \( h(u) \) is uniformly bounded, since we have assumed that \( d_A \) is bounded and continuous. Hence, since \( g \) bounded \( \Longrightarrow g_u \) bounded, it follows that \( (h(u) - H)g_u \) tends to zero in \( \mathbb{P}^\alpha \)-probability and also in \( L^1 \). Using the Markov property, we then deduce that on \( \{u \leq T_u^A\} \),

\[
\lim_{u \to \infty} \mathbb{E}^\alpha_{u, \mathcal{F}_u} \left[ (h(u) - H) g_u(V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] = H \cdot \lim_{u \to \infty} \mathbb{E}^\alpha_{X_u, V_u} \left[ g_u(V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] \quad \mathbb{P}^\alpha \text{-a.s.} \quad (7.11)
\]

**Step 2.** Next assume that either (i) or (ii) holds. Then we claim that for all \( x \in \mathbb{S}^{d-1}, v \in \mathbb{R}^d \),

\[
\lim_{u \to \infty} \mathbb{E}^\alpha_{x, v} \left[ g_u(V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] = \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \mathbb{E}^\alpha_y \left[ g(X_0, e^{\tilde{S}_1} X_1, e^{\tilde{S}_2} X_2, \ldots) \right] \sigma(dy, ds) \quad (7.12)
\]

We focus on the proof under the set of assumptions (ii), as the calculations needed under (i) are essentially identical, except simpler.

Thus assume (ii) holds and, for \( m \in \mathbb{N}_+ \), consider the decomposition

\[
\mathbb{E}^\alpha_{x, v} \left[ g_u(V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] = \mathbb{E}^\alpha_{x, v} \left[ (g - \mathcal{P}_m[g]) (V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] + \mathbb{E}^\alpha_{x, v} \left[ \mathcal{P}_m[g] (V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] - \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \mathbb{E}^\alpha_y \left[ g(X_0, e^{\tilde{S}_1} X_1, e^{\tilde{S}_2} X_2, \ldots) \right] \sigma(dy, ds)
\]

\[
\quad + \mathbb{E}^\alpha_{x, v} \left[ \mathcal{P}_m[g] (V_{j_u}^{(u)}, \ldots, V_{j_u}^{(u)}) \right] - \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \mathbb{E}^\alpha_y \left[ g(X_0, e^{\tilde{S}_1} X_1, e^{\tilde{S}_2} X_2, \ldots) \right] \sigma(dy, ds)
\]

\[
:= \mathbb{I}_1(u, m) + \mathbb{I}_2(u, m) + \mathbb{I}_3(u, m). \quad (7.13)
\]

Next we show that \( \mathbb{I}_1(u, m) + \mathbb{I}_3(u, m) \leq \Delta(m) \) for a sequence \( \Delta(m) \) which is independent of \( u \) and tends to zero as \( m \to \infty \).

First consider \( \mathbb{I}_1(u, m) \). Note by Lemma 3.15 that \( \mathbb{P}^\alpha(J_u - I_u > m) \to 0 \) as \( u \to \infty \). Thus it is sufficient to study the restriction to the set \( \{J_u - I_u \leq m\} \). Now let \( \{(M_n, Q_n) : n = 0, 1, \ldots\} \) be a process which is independent of \( \{(M_n, Q_n) : n = 0, 1, \ldots\} \) but sharing the same distribution function,
and let \( V_0^u \) denote the initial value corresponding to \( \{ (M_n^u, Q_n^u) \} \). Then using the \( \theta \)-Hölder continuity of \( g \) and the Markov property, together with the subadditivity of \( | \cdot |^\theta, \theta \leq 1 \), we obtain

\[
|I_1(u, m)| \leq \mathbb{E}_{x,v}^a \left[ \sum_{j=m}^{\infty} \left( V_{I_u}^{(u)} \right)^j \right] \leq \mathbb{E}_{x,v}^a \left[ \sum_{j=m}^{\infty} \mathbb{E}_{X_{I_u},\tilde{V}_{I_u}} \left[ a^{-\theta j} \left| M_j^2 \cdots M_1^2 \right| V_0^u + \sum_{k=1}^j M_j^k \cdots M_1^k Q_k^u \right]^\theta \right] \\
\leq \mathbb{E}_{x,v}^a \left[ \sum_{j=m}^{\infty} a^{-\theta j} B_1 \mathbb{E} \left[ \left| M_j \cdots M_1 \right|^\alpha \left( \left| M_j \cdots M_1 \right|^\theta + \sum_{k=1}^j \left| M_j \cdots M_{k+1} Q_k \right|^\theta \right) \right] \right].
\]

In the last line, we used the definition of the \( \alpha \)-shifted measure and set \( B_1 = \max_{x,y \in S_{+}^d-1} \left( r_\alpha(x)/r_\alpha(y) \right) \).

Now Corollary 4.6 in Buraczewski et al. (2014) gives that \( \mathbb{E} \left[ \left| M_j \cdots M_1 \right|^\theta \right] \leq B_2 \left( \lambda(\beta) \right)^j \) for all \( \beta \in \mathcal{D} \), where \( B_2 \) is a finite constant which is independent of \( j \). Thus we deduce that \( |I_1(u, m)| \) is bounded above by

\[
B_1 \sum_{j=m}^{\infty} a^{-\theta j} \left( \mathbb{E} \left[ \left| M_j \cdots M_1 \right|^\alpha \right] + \sum_{k=1}^j \mathbb{E} \left[ \left| M_j \cdots M_{k+1} \right|^\alpha \right] \mathbb{E} \left[ \left| Q_k \right|^\theta \right] \left| M_{k-1} \cdots M_1 \right|^\alpha \right) \\
\leq B_1 \sum_{j=m}^{\infty} \left( B_{\alpha+\theta} \frac{(\lambda(\alpha+\theta))}{a^{\theta}} \right)^j + j \sum_{k=1}^j \frac{\lambda(\alpha+\theta)-k\lambda(a-1)}{a^{\theta}} B_{\alpha+\theta} B_{\alpha+\theta} \mathbb{E} \left[ \left| M \right|^\alpha \left| Q \right|^\theta \right] \\
\leq B \sum_{j=m}^{\infty} (j+1) \frac{(\lambda(\alpha+\theta))}{a^{\theta}} := \Delta_1(m)
\]

for some finite constant \( B \). Recall that by assumption, \( \mathbb{E}[\left| M \right|^{\alpha+\theta} | Q |^\theta] \) is finite and \( \theta \log a > \lambda(\alpha+\theta) \leftrightarrow a^\theta > \lambda(\alpha+\theta) \). Thus \( \Delta_1(m) \to 0 \) as \( m \to \infty \).

Repeating the same argument for the term \( I_3(u, m) \) yields that \( |I_3(u, m)| \leq \Delta_2(m) \) where \( |\Delta_2(m)| \to 0 \) as \( m \to \infty \). [In contrast to the previous calculation, the terms involving \( Q_k \) now vanish.]

Next we show that \( \lim_{u \to \infty} \| I_2(u, m) \| = 0 \) for all \( m \in \mathbb{N} \). To this end, define

\[
G_u(x, v) = \mathbb{E}_{x}^a \left[ \mathcal{P}_{[m]} g \left( \tilde{v}, a^{-1} \left( M_1 \tilde{v} + \frac{V_1^{(0)}}{v \varepsilon_u} \right), \ldots, a^{-m} \left( M_m \cdots M_1 \tilde{v} + \frac{V_m^{(0)}}{v \varepsilon_u} \right) \right) \right];
\]

\[
G(x, v) = \mathbb{E}_{x}^a \left[ \mathcal{P}_{[m]} g \left( \tilde{v}, a^{-1} M_1 \tilde{v}, \ldots, a^{-m} M_m \cdots M_1 \tilde{v} \right) \right],
\]

where \( V_k^{(0)} \) is defined as in (3.52) for \( k \in \mathbb{N} \). Then the \( \theta \)-Hölder continuity of \( g \) gives that

\[
\lim_{u \to \infty} \sup \left\{ \left| G_u(x, v) - G(x, v) \right| : x \in S_{+}^{d-1}, v \in \mathbb{R}_{+}^d \setminus B_1(0) \right\} = 0.
\]

Now by Lemma 3.12, \( \lim_{u \to \infty} X_{I_u}/\tilde{V}_{I_u} \to 1 \) in \( \mathbb{P}_{x,v}^0 \)-probability. Moreover, under \( \mathbb{P}_{x,v}^0 \), we have by Melfi’s nonlinear renewal theorem (Theorem 3.10) that \( \varepsilon_u^{-1} \left( \tilde{V}_{I_u}, \log |\tilde{V}_{I_u}| \right) \) converges in law to a random variable \( (X, S) \), say, having the distribution \( \phi \). Then by the continuous mapping theorem and Slutsky’s theorem,

\[
(X_{I_u}, \varepsilon_u^{-1} V_{I_u}) = \left( \frac{X_{I_u}}{\tilde{V}_{I_u}}, \frac{V_{I_u}}{\varepsilon_u} \right) \Rightarrow (X, e^S X) \quad \text{as} \quad u \to \infty.
\]

In this notation, \( \mathbb{E}_{x,v}^a[G(X, e^S X)] = \int_{\mathbb{R}^+} \mathbb{E}_{y}^a \left[ \mathcal{P}_{[m]} g \left( X_0, e^S X_1, \ldots, e^S X_m \right) \right] \phi(dy, ds) \). Then, using the Markov property and the boundedness and continuity of \( G \),

\[
\lim_{u \to \infty} \| I_2(u, m) \| = \lim_{u \to \infty} \mathbb{E}_{x,v}^a \left[ G_u \left( X_{I_u}, \frac{V_{I_u}}{\varepsilon_u} \right) - G \left( X_{I_u}, \frac{V_{I_u}}{\varepsilon_u} \right) \right] \\
+ \lim_{u \to \infty} \mathbb{E}_{x,v}^a \left[ G \left( X_{I_u}, \frac{V_{I_u}}{\varepsilon_u} \right) \right] - \mathbb{E}_{x}^a \left[ G(X, e^S X) \right] = 0.
\]
Consider the decomposition bounded above by 2

Note that the same calculation proves (7.12) under the set of assumptions (i).

In conclusion, we have shown that for all $m \in \mathbb{N}_+$,

$$
\lim_{u \to \infty} \left| \mathbb{E}_x \left[ g_u \left( \Gamma_{I_u}, \ldots, \Gamma_{J_u} \right) \right] - \int_{\mathbb{S}_{t+1}^d \times \mathbb{R}_+} \mathbb{E}_y \left[ g \left( X_0, e^{S_1} X_1, e^{S_2} X_2, \ldots \right) \right] \, g(dy, ds) \right| \leq \Delta(m).
$$

Since $\Delta(m) \to 0$ as $m \to \infty$, we conclude that (7.12) holds.

**Step 3.** By combining (7.11) and (7.12), we obtain that

$$
\lim_{u \to \infty} \mathbb{1}_{\{n \leq T_\alpha^A\}} \mathbb{E}_x^o \left[ \mathcal{F}_u \mid \mathcal{F}_n \right] = H \int_{\mathbb{S}_{t+1}^d \times \mathbb{R}_+} \mathbb{E}_y^o \left[ g \left( X_0, e^{S_1} X_1, e^{S_2} X_2, \ldots \right) \right] \, g(dy, ds) \quad \mathbb{P}^\alpha\text{-a.s.}
$$

Finally, (7.8) is obtained by reasoning as in (4.16).

**Proof of Theorem 2.14.** It suffices to show that

$$
\limsup_{u \to \infty} u^\alpha \mathbb{E}_x \left[ \frac{1}{T_u^\alpha} \sum_{k=1}^{T_u^\alpha} g \left( \frac{1}{|V_k|} \right) \right] = \frac{1}{J_u} \mathbb{E}_x \left[ \frac{1}{J_u^\alpha} \sum_{k=1}^{J_u^\alpha} \mu_g \right] = 0.
$$

(7.17)

For simplicity, we introduce the shorthand notation

$$
\mu_g := \mathbb{E}(g(S_1)) \quad \text{and} \quad \Sigma^u = \sum_{k=1}^n g\left( \log |V_k| - \log |V_{k-1}| \right).
$$

Let $\{\varepsilon_n : u > 0\}$ be a sequence such that $\varepsilon_n = o(u)$ and $\varepsilon_n \uparrow \infty$ as $u \to \infty$. Set $\gamma_u = u - \varepsilon_u$ and $J_u = T_\gamma^\alpha$, and set $B_1 = \max_{x,y} (r_\alpha(x)/r_\alpha(y))$. Then from a change of measure, we infer that

$$
u^\alpha \mathbb{E}_x \left[ \frac{1}{T_u^\alpha} \sum_{k=1}^{T_u^\alpha} \mu_g \right] = r_\alpha(b) \mathbb{E}_x \left[ \frac{e^{-\alpha(S_u^\alpha - \log u)}}{r_\alpha(X_{T_u^\alpha})} \right] \left( \frac{1}{T_u^\alpha} \sum_{k=1}^{T_u^\alpha} \mu_g \right) \mathbb{E}_x \left[ \mathbb{1}_{\{T_u^\alpha < \tau\}} \right] = \mathbb{I}_1(u) + \mathbb{I}_2(u)
$$

First consider $\mathbb{I}_2(u)$. Note that

$$
\left| \frac{1}{J_u} \sum_{k=1}^{J_u} - \frac{1}{T_u^\alpha} \sum_{k=1}^{T_u^\alpha} \right| \leq \frac{1}{J_u} \left( \sum_{k=1}^{J_u} - T_u^\alpha \right) + \left( \frac{1}{J_u} \sum_{k=1}^{J_u} - \frac{1}{T_u^\alpha} \sum_{k=1}^{T_u^\alpha} \right) \leq 2 \left( \frac{T_u^\alpha - J_u^\alpha}{J_u} \right) \cdot |g|.
$$

By Lemma 3.15, $(J_u / \log u) \to (\lambda'(\alpha))^{-1}$ in $\mathbb{P}^\alpha$-probability and $((T_u^\alpha - J_u) / \log u) \to 0$ in $\mathbb{P}^\alpha$-probability. Hence the term inside the expectation of $\mathbb{I}_2(u)$ tends to zero in $\mathbb{P}^\alpha$-probability and, furthermore, is bounded above by $2 |g|$. Hence, $\limsup_{u \to \infty} \mathbb{I}_2(u) = 0$.

Next consider $\mathbb{I}_1(u)$. Here, our objective is to apply Fatou’s lemma and to observe that

$$
\limsup_{u \to \infty} \left| \frac{1}{J_u} \sum_{k=1}^{J_u} - \mu_g \right| = 0 \quad \mathbb{P}^\alpha\text{-a.s.}
$$

Consider the decomposition

$$
\left| \frac{1}{J_u} \sum_{k=1}^{J_u} - \mu_g \right| \leq \frac{1}{J_u} \sum_{k=1}^{J_u} g \left( \log \left( \frac{|V_k|}{|V_{k-1}|} \right) \right) - g(S_k - S_{k-1}) + \frac{1}{J_u} \sum_{k=1}^{J_u} g \left( S_k - S_{k-1} - \mu_g \right).
$$

By Lemma 3.15, $J_u \uparrow \infty$ a.s. as $u \to \infty$. The second term tends to zero $\mathbb{P}^\alpha\text{-a.s.}$, by Lemma 2.4. For the first term, use the Lipschitz continuity of $g$ to infer that for some finite constant $B_g$,

$$
\frac{1}{J_u} \sum_{k=1}^{J_u} g \left( \log \left( \frac{|V_k|}{|V_{k-1}|} \right) \right) - g(S_k - S_{k-1}) \leq B_g \sum_{k=1}^{J_u} \left( |\log |V_k| - \log |V_{k-1}| - (S_k - S_{k-1})| \right).
$$

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Also, it follows directly from the definitions (as given in (2.7) and (3.19)) that
\[
\log|V_k| - \log|V_{k-1}| - (S_k - S_{k-1}) := \log|Z_k| - \log|Z_{k-1}|.
\] (7.18)

Now by Lemma 3.6, $|Z_k|$ converges a.s. to the proper random variable $|Z|$ (and thus forms a Cauchy sequence). Then by Césaro’s theorem,
\[
\limsup_{u \to \infty} \frac{1}{J_u} \sum_{k=1}^{J_u} \left| \log|Z_k| - \log|Z_{k-1}| \right| = 0 \quad \text{a.s.,}
\] (7.19)
and we conclude that \( \limsup_{u \to \infty} I_1(u) = 0 \) \( \mathbb{P}^\alpha \)-a.s. This establishes (2.32).

**References**


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