Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

```plaintext
[ (customer.name, invoice.amount)
  | customer ← customers,
  invoice ← invoices,
  customer.cid = invoice.customer,
  invoice.due ≤ today ]
```

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \subseteq) \text{ means } f b \leq a \iff b \subseteq g a\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives

\[n \times k \leq m \iff n \leq m \div k\]

and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \rightarrow Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $id_X : X \rightarrow X$ for each $X$
- composition $f \cdot g : X \rightarrow Z$ of compatible arrows $g : X \rightarrow Y$ and $f : Y \rightarrow Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \rightarrow b$ iff $a \leq b$.

\[ \cdots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category $\mathbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
  h (m \otimes n) &= h m \oplus h n \\
  h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\mathbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A **functor** $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \text{id}_X = \text{id}_{F \text{X}}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, **forgetful** functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the **free** commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \cup, \emptyset)$$
$$\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories \( C, D \), and functors \( L : D \to C \) and \( R : C \to D \), adjunction means:

\[
[L X, Y] : C(L X, Y) \simeq D(X, R Y) : [R Y, X]
\]

A familiar example is given by currying:

\[
[- \times P] : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^o
\]

Hence definitions and properties of \( \text{apply} = \text{uncurry id}_{Y^P} : Y^P \times P \to Y \)
7. Products and coproducts

\[
\begin{array}{c}
\text{Set} \quad \perp \quad \text{Set}^2 \\
\Delta \quad \downarrow \quad \Delta
\end{array}
\]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork id}_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{ccc}
\text{CMon} & \downarrow & \text{Set} \\
\xRightarrow{\text{Free}} & & \xLeftarrow{\text{Set}} \\
\xRightarrow{U} & & \xLeftarrow{\text{with}} \left[ - \right]: \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
\xRightarrow{\approx Set}(A, U (M, \otimes, \epsilon)) & : & \left[ - \right]
\end{array}
\]

Unit and counit:

\[
\begin{align*}
\text{single } A &= \left[ \text{id}_{\text{Free } A} \right]: A \to U (\text{Free } A) \\
\text{reduce } M &= \left[ \text{id}_M \right]: \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h: \text{Free } A \to M \) and \( f: A \to U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>$(\mathbb{N}, 0, +)$</td>
<td>$\langle a \rangle \rightarrow 1$</td>
</tr>
<tr>
<td>sum</td>
<td>$(\mathbb{R}, 0, +)$</td>
<td>$\langle a \rangle \rightarrow a$</td>
</tr>
<tr>
<td>max</td>
<td>$(\mathbb{Z}, \minBound, \max)$</td>
<td>$\langle a \rangle \rightarrow a$</td>
</tr>
<tr>
<td>min</td>
<td>$(\mathbb{Z}, \maxBound, \min)$</td>
<td>$\langle a \rangle \rightarrow a$</td>
</tr>
<tr>
<td>all</td>
<td>$(\mathbb{B}, \text{True}, \land)$</td>
<td>$\langle a \rangle \rightarrow a$</td>
</tr>
<tr>
<td>any</td>
<td>$(\mathbb{B}, \text{False}, \lor)$</td>
<td>$\langle a \rangle \rightarrow a$</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

$$\text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A$$

$$\text{guard } p \ a = \text{if } p \ a \text{ then } \langle a \rangle \text{ else } \emptyset$$

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \to T A \\
\eta A & = [ id_A ] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction. The laws of exponents arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \xrightarrow{J} \text{Set} \xleftarrow{E} \text{Set}
\]

where \( J \) embeds, and \( E \, R : A \to \text{Set} \, B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} \,(K \times V) \approx \text{Map} \, K \,(\text{Bag} \, V)
\]

Together, \text{index} and \text{merge} give efficient relational joins:

\[
x_f \bowtie g \, y = \text{flatten} \,(\text{Map} \, K \, cp \,(\text{merge} \,(\text{groupBy} \, f \, x, \text{groupBy} \, g \, y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} \, V \to \text{Map} \, K \,(\text{Bag} \, V)
\]

\[
\text{flatten} \quad : \text{Map} \, K \,(\text{Bag} \, V) \to \text{Bag} \, V
\]
13. Pointed sets and finite maps

Model \textit{finite maps} \( \text{Map}_* \) not as partial functions, but \textit{total} functions to a \textit{pointed} codomain \((A, a)\), i.e. a set \( A \) with a distinguished element \( a : A \).

Pointed sets and point-preserving functions form a category \( \text{Set}_* \).

There is an adjunction to \( \text{Set} \), via

\[
\begin{array}{ccccc}
\text{Set}_* & 
\xrightarrow{\text{Maybe}} & 
\downarrow & 
\xleftarrow{U} & 
\text{Set} \\
\end{array}
\]

where \( \text{Maybe} \ A \simeq 1 + A \) adds a point, and \( U (A, a) = A \) discards it.

In particular, \((\text{Bag} \ A, \emptyset)\) is a pointed set. Moreover, \( \text{Bag} f \) is point-preserving, so we get a functor \( \text{Bag}_* : \text{Set} \to \text{Set}_* \).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta \ a = \lambda k \to a : A \to \text{Map} \ K \ A$$

in general yields an infinite map.

However, finite maps are a \textit{graded monad}*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_{\epsilon} X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

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