Relational algebra by way of adjunctions
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Publication date:
2016

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

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DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\left[ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \\
\text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} \right]
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\]

means \(f b \leq a \iff b \sqsubseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A *category* $\mathbf{C}$ consists of

- a set* $\mathcal{C}$ of *objects*,
- a set* $\mathcal{C}(X, Y)$ of *arrows* $X \to Y$ for each $X, Y : |\mathcal{C}|$,
- *identity* arrows $\text{id}_X : X \to X$ for each $X$,
- *composition* $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category **CMon** has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
  h (m \otimes n) &= h m \oplus h n \\
  h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category **Set** has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \ id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : CMon \to Set$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $Free : Set \to CMon$ generates the free commutative monoid (ie bags) on a set of elements:

$$Free A = (Bag A, \cup, \emptyset)$$
$$Free (f : A \to B) = map f : Bag A \to Bag B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $C, D,$ and functors $L : D \to C$ and $R : C \to D,$ adjunction $\downarrow \Downarrow \Downarrow$ means $[\cdot] : C(L X, Y) \simeq D(X, R Y) : [\cdot].$

A familiar example is given by currying:

$$\downarrow \Downarrow \Downarrow$$

hence definitions and properties of $apply = uncurry \ id_{Y^P} : Y^P \times P \to Y.$
7. Products and coproducts

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork id}_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{c}
\text{CMon} \\
\downarrow \hspace{1cm} \\
\text{Set}
\end{array}
\xleftarrow{\text{Free}}
\xrightarrow{U}

\text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U (M, \otimes, \epsilon)) : [-]

Unit and counit:

\[
\begin{align*}
\text{single } A & = [id_{\text{Free } A}] : A \to U (\text{Free } A) \\
\text{reduce } M & = [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>( \mathbb{N}, 0, + )</td>
<td>( { a } \rightarrow 1 )</td>
</tr>
<tr>
<td>sum</td>
<td>( \mathbb{R}, 0, + )</td>
<td>( { a } \rightarrow a )</td>
</tr>
<tr>
<td>max</td>
<td>( \mathbb{Z}, \text{minBound}, \text{max} )</td>
<td>( { a } \rightarrow a )</td>
</tr>
<tr>
<td>min</td>
<td>( \mathbb{Z}, \text{maxBound}, \text{min} )</td>
<td>( { a } \rightarrow a )</td>
</tr>
<tr>
<td>all</td>
<td>( \mathbb{B}, \text{True}, \land )</td>
<td>( { a } \rightarrow a )</td>
</tr>
<tr>
<td>any</td>
<td>( \mathbb{B}, \text{False}, \lor )</td>
<td>( { a } \rightarrow a )</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
\text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
\text{guard } p \ a = \text{if } p \ a \text{ then } \{ a \} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \( \mathbb{B} = 1 + 1 \)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag } A) \to \text{Bag } A \\
\text{single} & : A \to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f a b \mid a \leftarrow x, b \leftarrow g a \}\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R [id_A] L : T (T A) \to T A \\
\eta A & = [id_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \xrightarrow{J} \downarrow \text{Set} \xleftarrow{E} \text{Set}
\]

where \(J\) embeds, and \(E \ R : A \rightarrow \text{Set} \ B\) for \(R : A \sim B\).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} \ (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V)
\]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[
x f \bowtie g y = \text{flatten} \ (\text{Map} \ K \ cp \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y)))
\]

\[
\text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} \ V \rightarrow \text{Map} \ K \ (\text{Bag} \ V)
\]

\[
\text{flatten} : \text{Map} \ K \ (\text{Bag} \ V) \rightarrow \text{Bag} \ V
\]
13. Pointed sets and finite maps

Model *finite maps* \( \text{Map}_* \) not as partial functions, but *total* functions to a *pointed* codomain \((A, a)\), i.e. a set \(A\) with a distinguished element \(a : A\).

Pointed sets and point-preserving functions form a category \( \text{Set}_* \).

There is an adjunction to \( \text{Set} \), via

\[
\begin{array}{ccc}
\text{Set}_* & \overset{\perp}{\longrightarrow} & \text{Set} \\
\downarrow & \downarrow \\
\text{Maybe} & \rightarrow & \text{U}
\end{array}
\]

where \( \text{Maybe} A \simeq 1 + A \) adds a point, and \( \text{U} (A, a) = A \) discards it.

In particular, \((\text{Bag} A, \emptyset)\) is a pointed set. Moreover, \(\text{Bag} f\) is point-preserving, so we get a functor \(\text{Bag}_*: \text{Set} \to \text{Set}_*\).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid $$(M, \otimes, \epsilon)$$,

$$\mu X : T_m (T_n X) \rightarrow T_{m \otimes n} X$$
$$\eta X : X \rightarrow T\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid $$(\mathbb{K}, \times, 1)$$ of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.