Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\begin{array}{l}
\left(\text{customer.name, invoice.amount}\right) \\
\mid \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer.cid = invoice.customer,} \\
\text{invoice.due} \leq \text{today}\end{array}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\]

means \(f b \leq a \iff b \sqsubseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

\[(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives
\(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $id_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category \textbf{CMon} has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h: (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \textbf{Set} has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor \( F : C \to D \) is an operation on both objects and arrows, preserving the structure: \( F f : F X \to F Y \) when \( f : X \to Y \), and

\[
F id_X = id_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, forgetful functor \( U : \text{CMon} \to \text{Set} \):

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, Free : \text{Set} \to \text{CMon} generates the free commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathbf{C}, \mathbf{D}$, and functors $L : \mathbf{D} \to \mathbf{C}$ and $R : \mathbf{C} \to \mathbf{D}$, adjunction $\mathbf{C} \perp \mathbf{D}$ means $[-] : \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-]$.

A familiar example is given by currying:

$\mathbf{Set} \perp \mathbf{Set}$ with $curry : \mathbf{Set}(X \times P, Y) \simeq \mathbf{Set}(X, Y^P) : curry^\circ$

hence definitions and properties of $apply = uncurry \ id_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{align*}
\text{Set} & \xrightarrow{\bot} \text{Set}^2 \xrightarrow{\Delta} \text{Set} \\
\text{Set}^2 & \xrightarrow{\bot} \text{Set} \xrightarrow{\times} \text{Set}
\end{align*}
\]

with

\[
\begin{align*}
\text{fork} &: \text{Set}^2(\Delta A, (B, C)) \cong \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ &: \text{Set}(A + B, C) \cong \text{Set}^2((A, B), \Delta C) : \text{junc}
\end{align*}
\]

hence

\[
\begin{align*}
dup &= \text{fork } id_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) &= \text{fork}^\circ id_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\end{align*}
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{tikzcd}
\text{CMon} & \text{Set} \\
\downarrow \text{Free} & \downarrow \text{U} \\
\end{tikzcd}
\]

with \([-\cdot] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, \text{U} (M, \otimes, \epsilon)) : [-\cdot]

Unit and counit:

\[
\begin{align*}
\text{single } A &= [\text{id}_{\text{Free } A}] : A \to \text{U} (\text{Free } A) \\
\text{reduce } M &= [\text{id}_M] : \text{Free} (\text{U } M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \(h : \text{Free } A \to M\) and \(f : A \to \text{U } M = M\),

\[
h = \text{reduce } M \cdot \text{Free } f \iff \text{U } h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
## 9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
\text{guard} : (A ightarrow \mathbb{B}) \rightarrow \text{Bag } A ightarrow \text{Bag } A
\]

\[
\text{guard } p \ a = \text{if } p \ a \ \text{then } \{a\} \ \text{else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad (Bag, union, single) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag } A) \to \text{Bag } A \\
\text{single} & : A \to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \( \{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \} \).

In fact, for any adjunction \( L \dashv R \) between \( C \) and \( D \), we get a monad \( (T, \mu, \eta) \) on \( D \), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \ [ id_A ] \ L : T (T A) \to T A \\
\eta A & = [ id_A ] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction.

The laws of exponents arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 V & \simeq 1 \\
\text{Map } 1 V & \simeq V \\
\text{Map } (K_1 + K_2) V & \simeq \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \simeq \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \simeq 1 \\
\text{Map } K (V_1 \times V_2) & \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\begin{array}{c}
\text{Rel} \\ J \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\text{Set} \\
\leftarrow \\
\text{E} \\
\end{array}
\]

where \( J \) embeds, and \( E \) \( R : A \to \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V)
\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[
x f \Join g y = \text{flatten} (\text{Map} K \ cp (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V)
\]

\[
\text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V
\]
13. Pointed sets and finite maps

Model \textit{finite maps} \( \text{Map}_* \) not as partial functions, but \textit{total} functions to a \textit{pointed} codomain \((A, a)\), i.e. a set \( A \) with a distinguished element \( a : A \).

Pointed sets and point-preserving functions form a category \( \text{Set}_* \).

There is an adjunction to \( \text{Set} \), via

\[
\begin{array}{ccc}
\text{Set}_* & \perp & \text{Set} \\
\text{Maybe} & \nearrow & \searrow \\
\text{U} & & \\
\end{array}
\]

where \( \text{Maybe } A \cong 1 + A \) adds a point, and \( \text{U } (A, a) = A \) discards it.

In particular, \((\text{Bag } A, \emptyset)\) is a pointed set. Moreover, \( \text{Bag } f \) is point-preserving, so we get a functor \( \text{Bag}_*: \text{Set} \to \text{Set}_* \).

Indexing remains an isomorphism:

\[
\text{index : Bag}_* (K \times V) \cong \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_{\epsilon} X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((K, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

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