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A $c = 1$ phase transition in two-dimensional CDT/Horava–Lifshitz gravity?

J. Ambjørn\textsuperscript{a,}\textsuperscript{,b}, A. Görlich\textsuperscript{a,}\textsuperscript{,c},* J. Jurkiewicz\textsuperscript{c}, H. Zhang\textsuperscript{c}

\textsuperscript{a} The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen \O, Denmark
\textsuperscript{b} Institute for Mathematics, Astrophysics and Particle Physics (IMAPP), Radboud University Nijmegen, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands
\textsuperscript{c} Institute of Physics, Jagiellonian University, Lojasiewicza 11, 30-348 Krakow, Poland

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\textbf{A B S T R A C T}

We study matter with central charge $c > 1$ coupled to two-dimensional (2d) quantum gravity, here represented as causal dynamical triangulations (CDT). The 2d CDT is known to provide a regularization of (Euclidean) 2d Hořava–Lifshitz quantum gravity. The matter fields are massive Gaussian fields, where the mass is used to monitor the central charge $c$. Decreasing the mass we observe a higher order phase transition between an effective $c = 0$ theory and a theory where $c > 1$. In this sense the situation is somewhat similar to that observed for “standard” dynamical triangulations (DT) which provide a regularization of 2d quantum Liouville gravity. However, the geometric phase observed for $c > 1$ in CDT is very different from the corresponding phase observed for DT.

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slices at the discrete integer-labeled times then have the topology of a circle $S^1$. Equivalently, one can use the dual lattice with points in the centers of triangles connected by links, dual to the links of the triangles. In this paper we use this dual formulation.

Without matter fields the model can be solved analytically [12]. Let $\langle n(t) \rangle$ denote the average spatial volume measured at time $t$. If the time direction has length $L$ and the spacetime volume is $N$ we have $\langle n(t) \rangle = N/L$. The fluctuations around this average value can also be calculated analytically. If we couple matter fields to the geometry, one observes the same (trivial) picture as long as the central charge $c \leq 1$ for the matter fields [13]. However, if $c > 1$ one observes a change in the behavior of the universe [10,11].

If one looks at the distribution $n(t)$ in a single $(1+1)$-dimensional universe generated by Monte Carlo simulations one observes a “blob” and a “stalk”. In the stalk $n(t)$ is of the order of the cut-off. In the computer simulations we do not allow $n(t)$ to shrink to zero which would result in disconnected universes and we thus put in a lower cut off $n(t) = 2$. In the blob we have large $n(t)$’s and the average time extent of the blob scales as $N^{1/3}$, independent of $L$ if $L$ is larger than the size of the blob. As a function of computer time the “center of volume” of the blob is performing a random walk in the periodic time direction and to measure average properties of the blob we have to break the translation symmetry in our periodic discrete time. For each configuration we define $t = 0$ as the “center of volume” of the blob.  In this way one can obtain the average spatial volume distribution of the blob with high accuracy:

$$\langle n(t) \rangle = \frac{2}{\pi} N^{1-1/3} \cos \left( \frac{\alpha t}{N^{1/3}} \right), \quad |t| < \frac{\pi N^{1/3}}{2\alpha},$$

with $\alpha$ being a constant which depends on the central charge $c > 1$ of the matter fields, typically growing with $c$ [11].

The scaling of the blob as a function of the size $N$ is precisely what one expects for a (deformed) sphere $S^3$, $t$ being the distance from equator and we thus say that the Hausdorff dimension of the average two-dimension graph representing the blob is $D_H = 3$.

2. The model

A massless Gaussian field has central charge $c = 1$. Thus $d$ Gaussian fields have central charge $d$. In this paper we couple $d$ Gaussian fields to the geometry using the CDT model. The scalar fields $\phi^\mu, \mu = 1, \ldots, d$, are located at the vertices of the dual lattice. The combined system of geometry and matter is then a statistical model described by the partition function

$$Z = \sum_{T} \frac{1}{S_T} e^{-\lambda N_T} \int \prod_{i,j} d\phi^\mu e^{-S_{\text{measure}}(\phi^\mu, m)}$$

where $\lambda$ is a cosmological constant, $N_T$ is the number of vertices in the graph dual to triangulation $T$ and $S_T$ is a symmetry factor of the graph (the order of the automorphism group of the graph). The Gaussian measure (or action) $S_{\text{measure}}$ is defined as

$$S_{\text{measure}}(\phi^\mu, m) = \frac{1}{2} \sum_{i,j,\mu} (\phi^\mu_i - \phi^\mu_j)^2 + m^2 \sum_{i,\mu} (\phi^\mu_i)^2,$$
3. Mass dependence of the volume profiles

3.1. Small masses

For a small mass $m^2$ the average profile of spatial volumes $\langle n(t, m^2) \rangle$ contains a central blob where $\langle n(t, m^2) \rangle \gg 2$ and a stall where $\langle n(t, m^2) \rangle$ is of the cut-off size 2, as illustrated in Fig. 1 for a single configuration $n(t, m)$. For $t$ in the blob range, $\langle n(t, m^2) \rangle$ scales with $N$ in a way consistent with a Hausdorff dimension $D_H = 3$, i.e. the time extent of the blob scales as $N^{1/3}$. This means we can find a distribution $\rho(\tau, m^2) = N^{1/D_H} n(t, m^2)$, independent of $N$, plotted as a function of the scaled time variable $\tau = t/N^{1/D_H}$. The requirement of a scaling function $\rho(\tau, m^2)$ for different spacetime volumes $N$ determines $D_H = 3$ with good precision for all small values of $m^2$.

However, the universality is even larger. For small masses all distributions $\langle n(t, m^2) \rangle$ can be made to coincide if we, rather than scaling the time as $\tau = t/N^{1/2}$, define a rescaled time which depends on the mass: $\tilde{\tau} = \alpha(m^2) \tau$ and redefine the height of distribution accordingly as $\tilde{\rho}(\tilde{\tau}) = (\alpha(m^2))^{-1} N^{1/2} \langle n(t, m^2) \rangle$. A comparison of the rescaled distributions $\tilde{\rho}(\tilde{\tau})$ for $m^2 \in [0.01, 0.09]$ is presented in Fig. 2. The left curves are obtained by keeping the time variable $\tau$ unchanged but rescaling the maximum height of the curves to the $m^2 = 0$ curve (which is equivalent to multiplying $\rho(\tau, m^2)$ with $\alpha(0)/\alpha(m^2)$), provided a universal $\tilde{\rho}(\tilde{\tau})$ exists. The right curves are then obtained by rescaling $\tau$ to $\tilde{\tau}$ for the various curves, and in this way determining $\alpha(m^2)/\alpha(0)$ as the value leading to maximal overlap with the $m^2 = 0$ curve. It thus follows from (1) that

$$\tilde{\rho}(\tilde{\tau}) = \frac{2}{\pi} \cos^2 \tilde{\tau}, \quad \tilde{\tau} \in [-\pi/2, \pi/2].$$

The $\alpha$ values drop for larger mass (see Fig. 4 for the plot of $\alpha(m^2)/\alpha(0)$), implying that the blob gets broader when expressed in the unscaled time-variable. However, using the rescaled variable $\tilde{\tau}$ we can talk about one universal scaling distribution (4) of spatial volumes in the "blob" phase, independent of the mass for $m^2 \in [0, 0.09]$.

3.2. Large masses

The behavior is different for masses $m^2 \geq 0.15$. As was explained above we expect for large masses that $\langle n(t, m^2) \rangle$ will be qualitatively similar to the pure gravity case, where it is known analytically that any scaling should correspond to a Hausdorff dimension $D_H = 2$. In our approach we use the same method to center the volume of individual configurations as we used in the case where we observed a genuine blob (see footnote 2). As a consequence we see an artificial maximum around time $t = 0$, as already mentioned in footnote 3. The stall is absent, and the distributions have triangular shapes, with the height depending on the assumed period $L$ as $1/L$. Thus

$$\langle n(t, m^2) \rangle = \frac{N}{L} f(t/L, m^2), \quad \int_{-1/2}^{1/2} f(x, m^2) dx \approx 1. \quad (5)$$

If we want to look for scaling behavior of $\langle n(t, m^2) \rangle$ when changing $N$, we have to change the length of the time period $L$ simultaneously as $L \propto N^{1/2}$ since there is no stall. For the choice $L = \sqrt{N}$ Eq. (5) reads

$$\langle n(t, m^2) \rangle = N^{1/2} f(t/N^{1/2}, m^2). \quad (6)$$

For each value of $m^2 \geq 0.15$ we can extract a scaling function $f(t, \tau, m^2) = t/N^{1/2}$, by varying $N$. When comparing (6) with the general scaling form $N^{1-D_H} f(t/N^{1/D_H}, m^2)$ we find that the observed scaling indeed is compatible with $D_H = 2$.

In the same way as we did for the small masses, we now try to find a universal scaling of $\langle n(t, m^2) \rangle$ for all large masses. We construct the universal function in two steps, starting from the scaling functions $f(\tau, m^2)$, $\tau = t/N^{1/2}$, we already have available for each $m^2$. First we scale these functions such that they agree at $\tau = 0$ using $f(\tau, m^2_{\text{max}})$ as reference, i.e.

$$\tilde{f}(\tilde{\tau}, m^2) = \frac{f(0, m^2_{\text{max}})}{f(0, m^2)} f(\tau, m^2). \quad (7)$$

The result is shown in the left plot in Fig. 3. Then we try to rescale the time variable as we did for the small masses: $\tilde{\tau} = \beta(m^2) \tau/L$, where the function $\beta(m^2)$ is determined to ensure maximal overlap. This results in our universal scaling function

$$\tilde{f}(\tilde{\tau}, m^2) = \tilde{f}(\tilde{\tau}, m^2), \quad \tilde{\tau} = \beta(m^2) \tau \quad (8)$$

The result is shown in the right plot in Fig. 3 and the function $\beta(m^2)$ is shown in Fig. 4. Thus it makes sense to talk about one universal scaling function associated with $\langle n(t, m^2) \rangle$ also in the large mass regime, and this scaling function can be extracted from pure CDT without matter fields, which is the limit of $m^2 \rightarrow \infty$. From Fig. 4 it is seen that the function $\beta(m^2)$ is 1 for $m^2 > 0.18$, $m^2 = \infty$ thus effectively starts at $m^2 = 0.18$. In Fig. 4 we show values of $\alpha(m^2)/\alpha(0)$ and $\beta(m^2)/\beta(\infty)$ as functions of $m^2$. For $m^2 \in [0.10, 0.14]$ there is a cross over between the two well defined regimes corresponding to $D_H = 3$ and $D_H = 2$, respectively. In this range none of the fitting prescriptions described above works and using scaling arguments alone do not allow us to determine if there is a genuine phase transition or just a rapid cross over between the $D_H = 3$ and the $D_H = 2$ regions of $m^2$. 

Fig. 2. Scaling for small masses $m^2 \in [0, 0.09)$: the left plot shows the distributions $\langle n(t, m^2) \rangle$ plotted as functions of $\tau = t/N^{1/3}$. The right plot shows the same ratio $\langle n(t, m^2) \rangle$ plotted as a function of $\tilde{\tau} = \alpha(m^2) \tau$, where the factors $\alpha(m^2)$ are determined by maximizing the overlap between the various curves. The universal curve that emerges on the right plot is $\tilde{\rho}(\tilde{\tau})$ up to a normalization factor $\pi/2$ (see Eq. (4)).
4. Study of the phase transition

In order to study better the change from \( D_H = 3 \) to \( D_H = 2 \) we introduce the so-called volume–volume correlator \( \langle \text{corr}(\Delta) \rangle \), where \( \text{corr}(\Delta) \) is defined for individual configurations as

\[
\text{corr}(\Delta) = \sum_{i=1}^{L} n(t_i) n(t_i + \Delta)
\]

A great advantage of using the correlation function (9) is that one does not need to identify and to center the blob and it is well defined even if there is no blob.

A correlator similar to that defined by (9) was used in numerical studies of the scaling in three- and four-dimensional CDT [15]. We will measure \( \text{corr}(\Delta) \) at the maximal separation \( \Delta = L/2 \). In the small mass regime, where the blob is well localized we expect a behavior

\[
\langle \text{corr}(L/2) \rangle \approx 2hN
\]

where \( h \) is the average spatial volume of the time slices belonging to the stalk. As a consequence we expect in this mass regime that \( \langle \text{corr}(L/2) \rangle / N \approx 2h \), i.e. approximately both \( N \) and \( L \) independent.

In the large mass regime we expect a different behavior

\[
\langle \text{corr}(L/2) \rangle \approx N^2/L
\]

and consequently \( L/\langle \text{corr}(L/2) \rangle / N^2 \approx 1 \) should be \( N \) and \( L \) independent. In our analysis we fix the time period \( L \) and the spacetime volume \( N \) and measure the correlator as a function of \( m^2 \). In Fig. 5 we show the typical behavior of \( \langle \text{corr}(L/2) \rangle / N \) and \( L/\langle \text{corr}(L/2) \rangle / N^2 \) for \( L = 800 \) and a sequence of spacetime volumes \( N \). The plots illustrate the difference between the small and large mass behavior and indicate that there is a well defined transition between the two regimes.

To substantiate this we calculate the derivative \( d\langle \text{corr}(L/2) \rangle / dm^2 \). It has a clear peak growing with the size \( N \) of the system and thus signals a phase transition. In Fig. 6 we show the values of the numerically estimated derivative \( (1/N)\Delta \langle \text{corr}(L/2) \rangle / \Delta m^2 \) as a function of \( m^2 \) for \( L = 800 \) and for a sequence of spacetime volumes \( N \) (left plot) and the peak values of the estimated derivatives as a function of \( N \) (right plot).

The position of the maxima permits us to estimate the transition to be located at the critical mass \( m_c^2 \approx 0.135 \pm 0.005 \). A more precise determination of the critical mass \( m_c \) is difficult with the present numerical setup. The scaling of the maxima \( H(N) \) as a function of the spacetime volume \( N \) can be parametrized by

\[
H(N) \sim N^\alpha, \quad \alpha = 1.48 \pm 0.12
\]

strongly suggesting a higher order phase transition (the fit is presented on right plot in Fig. 6 as a red line).

5. Discussion and conclusion

We analyzed spatial volume distributions \( \langle n(t, m^2) \rangle \) for CDT geometries interacting with 4 massive scalar fields. There seem to be two regimes: a small mass regime with a universal distribution identical to the distribution obtained for massless fields, i.e. for a conformal field theory with central charge \( c = 4 \), containing a blob and a stalk, and with the blob scaling with Hausdorff dimension \( D_H = 3 \). The other regime where the masses are large also has a universal distribution scaling with \( D_H = 2 \) and the universal distribution is the one of pure gravity without any matter...
fields. Using the volume–volume correlator we located the critical mass $m_c^2$ where the transition between the two regime of different geometries takes place. The scaling of the derivative of the correlator at the critical mass $m_c^2$ as a function of system size suggests that the phase transition is of second or higher order.

We observe the same blob structure for any number $d > 1$ of massless Gaussian fields, as well as for multiple critical Ising spins corresponding to $c > 1$ coupled to CDT geometries. We have not observed the blobs for a single Ising spin, a single three-states Pott model or a single massless Gaussian field coupled to CDT geometries, systems which all have $c \leq 1$. Thus it is natural to conjecture that there is $c = 1$ barrier also in 2d CDT/Hořava–Lifshitz quantum gravity coupled to conformal field theories, and that it is a transition associated with this barrier that we observe by changing the mass of the four Gaussian fields.

It would be very interesting if one could solve the CDT model coupled to Gaussian fields analytically. Understanding the $c = 1$ barrier might help us to a better understanding of the $c = 1$ barrier in quantum Liouville gravity and understanding the formation of the blobs might help us to understand better the similar phenomenon in higher-dimensional CDT [16], where the appearance of the blob has been important in the attempts to define a continuum limit of lattice gravity [17,14].

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