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POLARIZED MORPHISMS BETWEEN ABELIAN VARIETIES

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Abstract. We study in this paper the Dynamical Manin-Mumford problem, focusing on the question of polarizability for endomorphisms of an abelian variety $A$ and on the action of a Frobenius and its Verschiebung on the diagonal subvariety of $A \times A$.

1. Introduction

We study in this work the role of polarizability for morphisms between abelian varieties in the Dynamical Manin-Mumford problem. Our results give also a way to strengthen the article [8].

Let us recall a few definitions: an endomorphism $\psi: X \to X$ of a projective variety is said to have a polarization if there exists an ample divisor $D$ on $X$ such that $\psi^* D \sim dD$ for some $d > 1$, where $\sim$ stands for linear equivalence. Another (equivalent) way of defining this notion is to use an ample line bundle $L$ over $X$ such that $\psi^* L = L \otimes d$. The integer $d$ is called the weight of the morphism $\psi$.

A subvariety $Y$ of $X$ is preperiodic under $\psi$ if there exists integers $m \geq 0$ and $s > 0$ such that $\psi^{m+s}(Y) = \psi^m(Y)$. We denote $\text{Prep}_\psi(X)$ the set of preperiodic points of $X$ under $\psi$. We will focus on the dynamical Manin-Mumford Conjecture 1.2.1 in [11]:

Conjecture 1.1. (Algebraic Dynamical Manin-Mumford) Let $\psi: X \to X$ be an endomorphism of a projective variety over a number field $k$ with a polarization, and let $Y$ be a subvariety of $X$. If $Y \cap \text{Prep}_\psi(X)$ is Zariski dense in $Y$, then $Y$ is a preperiodic subvariety.

This conjecture is very natural, it contains the classical Manin-Mumford conjecture proved by Raynaud in 1983: if one chooses $\psi = [n]$ and $X = A$ an abelian variety, then $\psi$ is polarized with weight $n^2$ and the preperiodic points are just torsion points.

One can find some examples of non-trivial endomorphisms where Conjecture 1.1 is true for the diagonal subvariety of a power of an abelian variety. Let $A$ be an abelian variety and $\mathcal{L}$ an ample symmetric line bundle. Consider the endomorphisms $\alpha$ and $\beta$ on $A^4$ given by $\alpha(x, y, z, t) = (x + z, y + t, x - z, y - t)$ and $\beta(x, y, z, t) = (x + y, x - y, z + t, z - t)$. If we let $\mathcal{L}_4 = p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L} \otimes p_4^* \mathcal{L}$, then we have $\alpha^* \mathcal{L}_4 = \beta^* \mathcal{L}_4 = \mathcal{L}_4^{\otimes 2}$, so the morphism $(\alpha, \beta)$ is polarized by $\mathcal{L}_4$. Of course one has $\alpha^2 = \beta^2$, so the diagonal of $A^4 \times A^4$ is actually preperiodic under the morphism $(\alpha, \beta)$.

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One can consult [1] for another situation, outside the abelian realm, where this conjecture is proved, namely on $\mathbb{P}^1 \times \mathbb{P}^1$ under a coordinatewise polynomial action.

D. Ghioca and T. Tucker found a family of counterexamples to this conjecture in [4]. They use squares of elliptic curves with complex multiplication. This was followed by the paper [5] where counterexamples of dimension four were constructed over squares of CM abelian surfaces, using complex multiplication on curves of genus 2 and some polarizability lemma. We show in this work a more general theorem, valid in greater dimension as well. The main result obtained is Theorem 1.2, which gives a general way of producing dynamical systems with particular intersection properties.

Let $k$ be a CM number field and $\overline{k}$ an algebraic closure. We will also denote the complex conjugation by a bar, but the context will always be clear. Let $\mathcal{O}_k$ be the ring of integers of $k$. Let $A$ be an abelian variety over $k$ with complex multiplication by $k$. Let $S_1$ be the set of ramified places of $k$, let $S_2$ be the set of places where $A$ has bad reduction and let $S_3$ be the set of places such that $F_a \neq V_a$ for all integers $n$, where $F_a$ is the Frobenius and $V_a$ the Verschiebung. We will denote by $S = S_1 \cup S_2 \cup S_3$ the reunion of these places.

**Theorem 1.2.** Let $A$ be an abelian variety defined over a number field $k$ that contains the field of complex multiplications $\text{End}_{\overline{k}}(A) \otimes \mathbb{Q}$ and such that $\mathcal{O}_k \subset \text{End}_{\overline{k}}(A)$. Let $p \notin S$. Let $F_p$ denote the Frobenius associated to $p$ and $V_p$ denote the Verschiebung associated to $F_p$. Then $F_p$ and $V_p$ are polarizable and the dynamical system $(A \times A, F_p \times V_p)$ together with the diagonal subvariety of $A \times A$ gives a counterexample to Conjecture 1.1.

This statement shows that one can actually use any CM abelian variety to construct a counterexample. Moreover, the way of producing these examples explains the origin of the first ones, and in a more geometric way.

There is a new version of Conjecture 1.1 that can be found in [4], where one takes into account the action on the tangent space of $T_{X,x}$ at preperiodic points $x$. More precisely one has:

**Conjecture 1.3.** (Ghioca, Tucker, Zhang) Let $\psi : X \to X$ be an endomorphism of a smooth projective variety over a number field $k$ with a polarization, and let $Y$ be a subvariety of $X$. Then $Y$ is preperiodic under $\psi$ if and only if there exists a Zariski dense subset of points $x \in Y \cap \text{Pre}_{\psi}(X)$ such that the tangent subspace of $Y$ at $x$ is preperiodic under the induced action of $\psi$ on the Grassmanian $\text{Gr}_{\dim(Y)}(T_{X,x})$.

Theorem 2.1 of [4] implies that Conjecture 1.3 holds when $\psi$ is a group endomorphism. Hence our constructions provide examples of subvarieties $Y$ containing a Zariski dense set of preperiodic points $x$ but such that $T_{Y,x}$ is not preperiodic under the action of $\psi$.

Using Lattès maps, one can transport these constructions on $\mathbb{P}^1 \times \mathbb{P}^1$, hence, using the Segre embedding, also on $\mathbb{P}^3$. A good question for future work would then be: what is the situation on $\mathbb{P}^2$?

In the next section we give the proof of Theorem 1.2. Then, as previously stated in [8], [2] and [9], obtaining polarizability can be difficult and is linked to arithmetic properties of the field $k$. Thus, we complete this work by giving some explicit polarizability criterions for morphisms between abelian varieties.
2. Abelian varieties and Frobenius maps

Let us start with the following lemma:

**Lemma 2.1.** Let $A$ be a projective variety over a field $k$, of dimension $g$ and let $\mathcal{L}$ be an ample line bundle. Let $f, j, h$ be three endomorphisms of $A$ such that $f \circ j = h$. Suppose $f^* \mathcal{L} = \mathcal{L}^\otimes d$ and $h^* \mathcal{L} = \mathcal{L}^\otimes n$. Then $j$ is polarized by $\mathcal{M} = \mathcal{L}^\otimes d$, $d \mid n$ and $j^* \mathcal{M} = \mathcal{M}_{\otimes}^\otimes d$.

**Proof.** We calculate $(f \circ j)^* \mathcal{L} = j^*(f^* \mathcal{L}) = j^*(\mathcal{L}^\otimes d)$, hence $j^*(\mathcal{L}^\otimes d) = \mathcal{L}^\otimes n$. Take the degree (associated to $\mathcal{L}$) to get: $\deg_{\mathcal{L}}(j) < \mathcal{L}^\otimes d >^g = \mathcal{L}^\otimes n >^g$, which gives $\deg_{\mathcal{L}}(j) d^g < \mathcal{L} >^g = n^g < \mathcal{L} >^g$, which is equivalent, as $\mathcal{L}$ is ample, to $\deg_{\mathcal{L}}(j) d^g = n^g$. So there exists an integer $m$ such that $n = dm$. This shows $j^*(\mathcal{L}^\otimes d) = (\mathcal{L}^\otimes d)^\otimes m$. $\square$

Now we will move on to the proof of Theorem [12]

**Proof.** One remark is that if $k$ is large enough, $S_2$ is empty. One finds in [2] pages 16-17, Proposition 3.2-3.3, that if $p \notin S_1 \cup S_2$, then the Frobenius $F_p$ is polarized by a symmetric line bundle $\mathcal{L}$.

Let $V_p$ denote the Verschiebung associated to $F_p$. Let $N = \text{Norm}(p)$. We know that $F_p^* (\mathcal{L}) = \mathcal{L}^\otimes N$ and that $[N]^* \mathcal{L} = \mathcal{L}^{N^2}$. As we have $F_p \circ V_p = [N]$, we can apply Proposition [2.1] to get $V_p^* (\mathcal{L}^\otimes N) = (\mathcal{L}^\otimes N)^\otimes N = \mathcal{L}^{\otimes N^2}$, hence $V_p$ is also polarizable, and with the same weight. Thus $F_p \times V_p$ is also polarized on $A \times A$.

Now, consider $\Delta = \{(P, P) \mid P \in A\}$, the diagonal subvariety of $A \times A$. This variety cannot be preperiodic under $\varphi = F_p \times V_p$, because it would imply $F_p^m = V_p^m$ for some positive integer $m$, which is impossible because it would imply $p \in S_2$. But $\Delta \cap \text{Prep}_\varphi(A \times A)$ is dense in $\Delta$ because it contains all torsion points of $\Delta$. $\square$

**Remark 2.2.** This theorem is a way to generalize the previous counterexamples of [4] and [8]. For example when an elliptic curve $E$ has a multiplication by $[i]$, the morphism $[2 + i]$ corresponds in fact to the Frobenius $F_{2-i}$, as is shown in [10], Proposition 4.2 page 122.

**Remark 2.3.** This theorem gives as a by-product an explanation about the fact that one needs to avoid the ramified places if one searches for polarizability in the CM case. For the number field $\mathbb{Q}[i]$, the discriminant is $-4$, so as $(1 + i) \mid (2)$, the morphism $[1 + i]$ will not be polarizable. On the contrary, as $(2 + i)$ and $(2)$ are coprime ideals, $[2 + i]$ is polarizable. This refines what is said in [9].

**Remark 2.4.** One can construct examples of polarized morphisms on $A \times A$ like in theorem [12] as soon as $\mathbb{Z} \subseteq \text{End}(A)$ and the Rosati involution is not trivial. We refer to [7] page 200, Theorem 2, for the classification of division algebras that can occur for $\text{End}\mathbb{Q}(A)$.

3. Polarizability criterions

A classical tool to get polarizability is the cube theorem. We refer to [9] for some formulas derived from this theorem and useful to get information on the weight of complex multiplications.

We give in this section other polarizability criterions, namely Proposition [5,1] for the action of the Rosati involution and Proposition [6,3] for the particular case of elliptic curves.
3.1. Rosati involution. Let $A$ be an abelian variety over a field $k$ and $\mathcal{L}$ be an ample line bundle. We denote by $\dagger$ the Rosati involution associated to $\mathcal{L}$. (See [6] page 137 for more details.) For any invertible line bundle $\mathcal{M}$, we let $\varphi_{\mathcal{M}}$ denote the classical application $a \mapsto t^*_a \mathcal{M} \otimes \mathcal{M}^{-1}$ from $A$ to $\text{Pic}^0(A)$.

**Proposition 3.1.** Let $A$ be an abelian variety, $\psi$ an endomorphism of $A$ and $\mathcal{L}$ an ample line bundle. Then $\psi$ is polarized by $\mathcal{L}$ with weight $d$ if and only if one has $\psi^\dagger \psi = [d]$, where $\dagger$ is associated to $\mathcal{L}$.

**Proof.** Let $\text{NS}_Q(A)$ be the Néron-Severi group of $A$ tensored by $Q$. Let $\text{End}_Q^{\dagger}(A)$ be the group of endomorphism fixed by $\dagger$. We thus have an isomorphism (see [6] page 137)

$$H : \text{NS}_Q(A) \to \text{End}_Q^{\dagger}(A)$$

$$\mathcal{M} \mapsto \varphi_{\mathcal{L}}^{-1} \circ \varphi_{\mathcal{M}}.$$

We then calculate that $H(\psi^* \mathcal{L}) = \psi^\dagger \psi$ and $H(\mathcal{L}^\otimes d) = [d]$, so we can express the polarizability condition $\psi^* \mathcal{L} = \mathcal{L}^\otimes d$ by $\psi^\dagger \psi = [d]$. $\square$

**Remark 3.2.** If $A$ has complex multiplication, then one has $\psi^\dagger = \overline{\psi}$. Hence, to study Conjecture [1] as we want to find endomorphisms $\psi$ such that for all integers $m \geq 1$, $\psi^m \neq (\psi^\dagger)^m$, in the CM case it boils down to finding a number $\alpha$ such that $\alpha\overline{\alpha} \in \mathbb{Z}$, $\alpha \notin \mathbb{Z}$ and $\alpha/\overline{\alpha}$ is not a root of unity.

**Application:** back to multiplication by $1 + \zeta_5$ not polarized by $\Theta$. Thanks to this Rosati action, we can add a remark to one of the examples of [3] where the morphism $[1 + \zeta_5]$ is not polarized by the divisor $\Theta$ on the jacobian of a genus 2 curve. But now a little calculation in the particular example of $z = 4 + 3\zeta_5 + 12\zeta_5^2$ gives $z\overline{z} = 121$ and $z/\overline{z}$ is not a root of unity. Hence by Proposition 3.1 we get that $[z]$ is polarized by $\Theta$.

3.2. Elliptic curves. In the particular case of elliptic curves, one can get a precise condition for polarizability. This is due to the fact that the variety is simple on the one hand and the fact that the divisor support is just a point on the other hand.

**Proposition 3.3.** Let $E$ be an elliptic curve over a field $k$ of characteristic zero and $f$ be an isogeny of $E$ of degree $d$. We denote by $E[2]$ the group of 2-torsion points over $k$. Then we have

$$\left(\text{Card}(E[2] \cap \text{Ker}(f)) \neq 2\right) \iff \left(f^*(O) \sim d(O)\right).$$

**Proof.** Let $H = E[2] \cap \text{Ker}(f)$ and $G = \text{Ker}(f) \setminus H$. Let $m = \text{Card}(H)$. We know that $m \in \{1, 2, 4\}$. Let us choose any short Weierstrass model $y^2 = x^3 + ax + b$. We calculate:

$$f^*(O) = \sum_{P \in \text{Ker}(f)} (P) = \sum_{P \in H} (P) + \sum_{P \in G} (P) = \sum_{P \in H} (P) + \sum_{P \in G/\pm 1} ((P) + (-P)),$$

and as $\text{div}(x - x(P)) = (P) + (-P) - 2(O)$, we have

$$\sum_{P \in G/\pm 1} ((P) + (-P)) \sim 2\left(\frac{d - m}{2}\right)(O),$$

thus $f^*(O) \sim \sum_{P \in H} (P) + (d - m)(O)$. Then we have three cases:
• if \( m = 1 \), then \( H = \{ O \} \) and \( f^*(O) \sim (O) + (d - 1)(O) \sim d(O) \),
• if \( m = 2 \), then \( H = \{ O, P \} \) and \( f^*(O) \sim (O) + (P) + (d - 2)(O) \sim (P) + (d - 1)(O) \), then the divisor \( D = (P) - (O) \) is of degree 0 but is not the divisor of a rational function on \( E \) because \( P - O \neq O \),
• if \( m = 4 \), then \( H = \{ O, P_1, P_2, P_1 + P_2 \} \) and \( f^*(O) \sim (O) + (P_1) + (P_2) + (P_1 + P_2) + (d - 4)(O) \sim d(O) \) because \( \text{div}(y) = (P_1) + (P_2) + (P_1 + P_2) - 3(O) \).

\[ \square \]

**Remark 3.4.** Let \( E \) be the elliptic curve with affine model \( y^2 = x^3 + x \). It has complex multiplication by \([i] : (x, y) \to (-x, iy)\). Then the isogeny \([1 + i] \) has degree 2, and as \([2] = [1 + i][1 - i] \), we get \( \text{Ker}[1 + i] \subset E[2] \), hence \( \text{Card}(E[2] \cap \text{Ker}[1 + i]) = 2 \), hence by Proposition 3.3 the morphism \([1 + i] \) is not polarized. But as shown in [8], the morphism \([2 + i] \) is polarized, another proof of this fact is that \([5] = [2 + i][2 - i] \), hence \( \text{Ker}[2 + i] \subset E[5] \), but \( E[5] \cap E[2] = O \), hence again by Proposition 3.3 the morphism \([2 + i] \) is polarized, with weight 5.

**Remark 3.5.** This result sharpens the remark made in the introduction of [9] concerning elliptic curves. See the remark 2.3 below for more details.

### 3.3. Explicit Lattès dynamical system.

Let \( E \) be the elliptic curve with affine model \( y^2 = x^3 + x \). It has complex multiplication by \([i] : (x, y) \to (-x, iy)\). Let \( \pi : E \to \mathbb{P}^1 \) be defined by \( \pi(x, y) = x \) and \( \pi(O) = \infty \). Then we get the following picture:

\[
\begin{array}{ccc}
E & \overset{[2+i]}{\longrightarrow} & E \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{P}^1 & \rightarrow & \mathbb{P}^1
\end{array}
\]

A direct calculation shows that the Lattès map \( \varphi \) can be expressed on the affine chart as

\[
\varphi(x) = \frac{(3 - 4i)x(3x^2 + 1 - 2i)^2}{(5x^2 + 1 + 2i)^2}.
\]

**Remark 3.6.** We thus get the \( x \)-coordinate of the four non-trivial \([2 + i] \)-torsion points: \( \pm x = \sqrt{\frac{\sqrt{5} - 1}{10}} + i \frac{\sqrt{10}}{\sqrt{\sqrt{5} - 1}} \).

For the Lattès map of \([2 - i] \), we get on the affine chart

\[
\psi(x) = \frac{(3 + 4i)x(3x^2 + 1 + 2i)^2}{(5x^2 + 1 - 2i)^2}.
\]

Now take a look at

\[
\delta : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\
(s, t) \mapsto (\varphi(s), \psi(t)).
\]

Then if \( D = \{ \infty \} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{ \infty \} \), we get \( \delta^*D \sim 5D \), hence \( \delta \) is polarized by \( D \) with weight 5. This gives an explicit counterexample to Conjecture 1.1 in the case of \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) as in the constructions of [4]. See [3] for Lattès dynamical systems.
References


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