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Polarized Morphisms Between Abelian Varieties

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Abstract. We study in this paper the Dynamical Manin-Mumford problem, focusing on the question of polarizability for endomorphisms of an abelian variety \( A \) and on the action of a Frobenius and its Verschiebung on the diagonal subvariety of \( A \times A \).

1. Introduction

We study in this work the role of polarizability for morphisms between abelian varieties in the Dynamical Manin-Mumford problem. Our results give also a way to strengthen the article [8].

Let us recall a few definitions: an endomorphism \( \psi : X \to X \) of a projective variety is said to have a polarization if there exists an ample divisor \( D \) on \( X \) such that \( \psi^*D \sim dD \) for some \( d > 1 \), where \( \sim \) stands for linear equivalence. Another (equivalent) way of defining this notion is to use an ample line bundle \( L \) over \( X \) such that \( \psi^*L = L \otimes d \). The integer \( d \) is called the weight of the morphism \( \psi \).

A subvariety \( Y \) of \( X \) is preperiodic under \( \psi \) if there exists integers \( m \geq 0 \) and \( s > 0 \) such that \( \psi^m + s(Y) = \psi^s(Y) \). We denote \( \text{Prep}_\psi(X) \) the set of preperiodic points of \( X \) under \( \psi \). We will focus on the dynamical Manin-Mumford Conjecture 1.2.1 in [11]:

Conjecture 1.1. (Algebraic Dynamical Manin-Mumford) Let \( \psi : X \to X \) be an endomorphism of a projective variety over a number field \( k \) with a polarization, and let \( Y \) be a subvariety of \( X \). If \( Y \cap \text{Prep}_\psi(X) \) is Zariski dense in \( Y \), then \( Y \) is a preperiodic subvariety.

This conjecture is very natural, it contains the classical Manin-Mumford conjecture proved by Raynaud in 1983: if one chooses \( \psi = [n] \) and \( X = A \) an abelian variety, then \( \psi \) is polarized with weight \( n^2 \) and the preperiodic points are just torsion points.

One can find some examples of non-trivial endomorphisms where Conjecture 1.1 is true for the diagonal subvariety of a power of an abelian variety. Let \( A \) be an abelian variety and \( L \) an ample symmetric line bundle. Consider the endomorphisms \( \alpha \) and \( \beta \) on \( A^4 \) given by \( \alpha(x, y, z, t) = (x + z, y + t, x - z, y - t) \) and \( \beta(x, y, z, t) = (x + y, x - y, z + t, z - t) \). If we let \( \mathcal{L}_4 = p_1^*L \otimes p_2^*L \otimes p_3^*L \otimes p_4^*L \), then we have \( \alpha^*\mathcal{L}_4 = \beta^*\mathcal{L}_4 = \mathcal{L}_4^{\otimes 2} \), so the morphism \( (\alpha, \beta) \) is polarized by \( \mathcal{L}_4 \). Of course one has \( \alpha^2 = \beta^2 \), so the diagonal of \( A^4 \times A^4 \) is actually preperiodic under the morphism \( (\alpha, \beta) \).
One can consult [1] for another situation, outside the abelian realm, where this conjecture is proved, namely on \( \mathbb{P}^1 \times \mathbb{P}^1 \) under a coordinatewise polynomial action.

D. Ghioca and T. Tucker found a family of counterexamples to this conjecture in [4]. They use squares of elliptic curves with complex multiplication. This was followed by the paper [8] where counterexamples of dimension four were constructed over squares of CM abelian surfaces, using complex multiplication on curves of genus 2 and some polarizability lemma. We show in this work a more general theorem, valid in greater dimension as well. The main result obtained is Theorem 1.2, which gives a general way of producing dynamical systems with particular intersection properties.

Let \( k \) be a CM number field and \( \overline{k} \) an algebraic closure. We will also denote the complex conjugation by a bar, but the context will always be clear. Let \( \mathcal{O}_k \) be the ring of integers of \( k \). Let \( A \) be an abelian variety defined over \( k \) with complex multiplication by \( k \). Let \( S_1 \) be the set of ramified places of \( k \), let \( S_2 \) be the set of places where \( A \) has bad reduction and let \( S_3 \) be the set of places such that \( F_n^a \neq V_n^a \) for all integers \( n \), where \( F_n^a \) is the Frobenius and \( V_n^a \) the Verschiebung. We will denote by \( S = S_1 \cup S_2 \cup S_3 \) the reunion of these places.

**Theorem 1.2.** Let \( A \) be an abelian variety defined over a number field \( k \) that contains the field of complex multiplications \( \text{End}_{\overline{k}}(A) \otimes \mathbb{Q} \) and such that \( \mathcal{O}_k \subset \text{End}_{\overline{k}}(A) \). Let \( p \notin S \). Let \( F_p \) denote the Frobenius associated to \( p \) and \( V_p \) denote the Verschiebung associated to \( F_p \). Then \( F_p \) and \( V_p \) are polarizable and the dynamical system \((A \times A, F_p \times V_p)\) together with the diagonal subvariety of \( A \times A \) gives a counterexample to Conjecture 1.1.

This statement shows that one can actually use any CM abelian variety to construct a counterexample. Moreover, the way of producing these examples explains the origin of the first ones, and in a more geometric way.

There is a new version of Conjecture 1.1 that can be found in [4], where one takes into account the action on the tangent space of \( T_{X,x} \) at preperiodic points \( x \). More precisely one has:

**Conjecture 1.3.** (Ghioca, Tucker, Zhang) Let \( \psi : X \to X \) be an endomorphism of a smooth projective variety over a number field \( k \) with a polarization, and let \( Y \) be a subvariety of \( X \). Then \( Y \) is preperiodic under \( \psi \) if and only if there exists a Zariski dense subset of points \( x \in Y \cap \text{Prep}_\psi(X) \) such that the tangent subspace of \( Y \) at \( x \) is preperiodic under the induced action of \( \psi \) on the Grassmanian \( \text{Gr}_{\dim(Y)}(T_{X,x}) \).

Theorem 2.1 of [4] implies that Conjecture 1.3 holds when \( \psi \) is a group endomorphism. Hence our constructions provide examples of subvarieties \( Y \) containing a Zariski dense set of preperiodic points \( x \) but such that \( T_{Y,x} \) is not preperiodic under the action of \( \psi \).

Using Lattès maps, one can transport these constructions on \( \mathbb{P}^1 \times \mathbb{P}^1 \), hence, using the Segre embedding, also on \( \mathbb{P}^3 \). A good question for future work would then be: what is the situation on \( \mathbb{P}^2 \)?

In the next section we give the proof of Theorem 1.2. Then, as previously stated in [8], [2] and [9], obtaining polarizability can be difficult and is linked to arithmetic properties of the field \( k \). Thus, we complete this work by giving some explicit polarizability criterions for morphisms between abelian varieties.
2. Abelian varieties and Frobenius maps

Let us start with the following lemma:

**Lemma 2.1.** Let $A$ be a projective variety over a field $k$, of dimension $g$ and let $L$ be an ample line bundle. Let $f, j, h$ be three endomorphisms of $A$ such that $f \circ j = h$. Suppose $f^*L = L^\otimes d$ and $h^*L = L^\otimes m$. Then $j$ is polarized by $M = L^\otimes d$, $d | n$ and $j^*M = M^\otimes m$.

**Proof.** We calculate $(f \circ j)^*L = j^*(f^*L) = j^*(L^\otimes d)$, hence $j^*(L^\otimes d) = L^\otimes n$. Take the degree (associated to $L$) to get: $\text{deg}(j) < L^\otimes d >^n = L^\otimes n >^g$, which gives $\text{deg}(j)^g < L >^g = n^g < L >^g$, which is equivalent, as $L$ is ample, to $\text{deg}(j)^g = n^g$. So there exists an integer $m$ such that $n = dm$. This shows $j^*(L^\otimes d) = (L^\otimes d)^\otimes m$. □

Now we will move on to the proof of Theorem [12]

**Proof.** One remark is that if $k$ is large enough, $S_2$ is empty. One finds in [2] pages 16-17, Proposition 3.2-3.3, that if $p \notin S_1 \cup S_2$, then the Frobenius $F_p$ is polarized by a symmetric line bundle $L$.

Let $V_p$ denote the Verschiebung associated to $F_p$. Let $N = \text{Norm}(p)$. We know that $F_p^*(L) = L^\otimes N$ and that $[N]^*L = L^\otimes N^2$. As we have $F_p \circ V_p = [N]$, we can apply Proposition [2.1] to get $V_p^*(L^\otimes N) = (L^\otimes N)^\otimes N = L^\otimes N^2$, hence $V_p$ is also polarizable, and with the same weight. Thus $F_p \times V_p$ is also polarized on $A \times A$.

Now, consider $\Delta = \{(P, P) \mid P \in A\}$, the diagonal subvariety of $A \times A$. This variety cannot be preperiodic under $\varphi = F_p \times V_p$, because it would imply $F_p^m = V_p^m$ for some positive integer $m$, which is impossible because it would imply $p \in S_3$. But $\Delta \cap \text{Pre}_{\varphi}(A \times A)$ is dense in $\Delta$ because it contains all torsion points of $\Delta$. □

**Remark 2.2.** This theorem is a way to generalize the previous counterexamples of [4] and [8]. For example when an elliptic curve $E$ has a multiplication by $[i]$, the morphism $[2 + i]$ corresponds in fact to the Frobenius $F_{(2-i)}$, as is shown in [10], Proposition 4.2 page 122.

**Remark 2.3.** This theorem gives as a by-product an explanation about the fact that one needs to avoid the ramified places if one searches for polarizability in the CM case. For the number field $\mathbb{Q}[i]$, the discriminant is $-4$, so as $(1 + i) | (2)$, the morphism $[1 + i]$ will not be polarizable. On the contrary, as $(2 + i)$ and (2) are coprime ideals, $[2 + i]$ is polarizable. This refines what is said in [9].

**Remark 2.4.** One can construct examples of polarized morphisms on $A \times A$ like in theorem [12] as soon as $Z \subseteq \text{End}(A)$ and the Rosati involution is not trivial. We refer to [7] page 200, Theorem 2, for the classification of division algebras that can occur for $\text{End}_\mathbb{Q}(A)$.

3. Polarizability criterions

A classical tool to get polarizability is the cube theorem. We refer to [8] for some formulas derived from this theorem and useful to get information on the weight of complex multiplications.

We give in this section other polarizability criterions, namely Proposition [5.1] for the action of the Rosati involution and Proposition [6.3] for the particular case of elliptic curves.
3.1. Rosati involution. Let $A$ be an abelian variety over a field $k$ and $\mathcal{L}$ be an ample line bundle. We denote by $\dagger$ the Rosati involution associated to $\mathcal{L}$. (See \cite{[1]} page 137 for more details.) For any invertible line bundle $\mathcal{M}$, we let $\varphi_\mathcal{M}$ denote the classical application $a \mapsto t_\mathcal{M}^* \mathcal{M} \otimes \mathcal{M}^{-1}$ from $A$ to $\Pic^0(A)$.

**Proposition 3.1.** Let $A$ be an abelian variety, $\psi$ an endomorphism of $A$ and $\mathcal{L}$ an ample line bundle. Then $\psi$ is polarized by $\mathcal{L}$ with weight $d$ if and only if one has $\psi^{\dagger} \psi = [d]$, where $\dagger$ is associated to $\mathcal{L}$.

**Proof.** Let $\NS_Q(A)$ be the Néron-Severi group of $A$ tensored by $Q$. Let $\End^1_Q(A)$ be the group of endomorphism fixed by $\dagger$. We thus have an isomorphism (see \cite{[1]} page 137)

$$H : \NS_Q(A) \rightarrow \End^1_Q(A)$$

$$\mathcal{M} \mapsto \varphi_\mathcal{M}^{-1} \circ \varphi_{\mathcal{M}}.$$ 

We then calculate that $H(\psi^{\dagger} \mathcal{L}) = \psi^{\dagger} \psi$ and $H(\mathcal{L}^{\otimes d}) = [d]$, so we can express the polarizability condition $\psi^{\dagger} \mathcal{L} = \mathcal{L}^{\otimes d}$ by $\psi^{\dagger} \psi = [d]$. 

**Remark 3.2.** If $A$ has complex multiplication, then one has $\psi^{\dagger} = \overline{\psi}$. Hence, to study Conjecture \cite{[1]}, as we want to find endomorphisms $\psi$ such that for all integers $m \geq 1$, $\psi^m \neq (\psi^{\dagger})^m$, in the CM case it boils down to finding a number $\alpha$ such that $\alpha \overline{\alpha} \in \mathbb{Z}$, $\alpha \notin \mathbb{Z}$ and $\alpha / \overline{\alpha}$ is not a root of unity.

**Application:** back to multiplication by $1 + \zeta_5$ not polarized by $\Theta$. Thanks to this Rosati action, we can add a remark to one of the examples of \cite{[5]} where the morphism $[1 + \zeta_5]$ is not polarized by the divisor $\Theta$ on the jacobian of a genus 2 curve. But now a little calculation in the particular example of $z = 4 + 3\zeta_5 + 12\zeta_5^2$ gives $z\overline{z} = 121$ and $z / \overline{z}$ is not a root of unity. Hence by Proposition 3.1 we get that $[z]$ is polarized by $\Theta$.

3.2. Elliptic curves. In the particular case of elliptic curves, one can get a precise condition for polarizability. This is due to the fact that the variety is simple on the one hand and the fact that the divisor support is just a point on the other hand.

**Proposition 3.3.** Let $E$ be an elliptic curve over a field $k$ of characteristic zero and $f$ be an isogeny of $E$ of degree $d$. We denote by $E[2]$ the group of 2-torsion points over $k$. Then we have

$$\left( \text{Card}(E[2] \cap \text{Ker}(f)) \neq 2 \right) \Leftrightarrow \left( f^*(O) \sim d(O) \right).$$

**Proof.** Let $H = E[2] \cap \text{Ker}(f)$ and $G = \text{Ker}(f) \setminus H$. Let $m = \text{Card}(H)$. We know that $m \in \{1, 2, 4\}$. Let us choose any short Weierstrass model $y^2 = x^3 + ax + b$. We calculate:

$$f^*(O) = \sum_{P \in \text{Ker}(f)} (P) = \sum_{P \in H} (P) + \sum_{P \in G} (P) = \sum_{P \in H} (P) + \sum_{P \in G / \{\pm 1\}} ((P) + (-P)),$$

and as $\text{div}(x - x(P)) = (P) + (-P) - 2(O)$, we have

$$\sum_{P \in G / \{\pm 1\}} ((P) + (-P)) \sim 2 \left( \frac{d - m}{2} \right)(O),$$

thus $f^*(O) \sim \sum_{P \in H} (P) + (d - m)(O)$. Then we have three cases:
• if \( m = 1 \), then \( H = \{ O \} \) and \( f^*(O) \sim (O) + (d - 1)(O) \sim d(O) \),
• if \( m = 2 \), then \( H = \{ O, P \} \) and \( f^*(O) \sim (O) + (P) + (d - 2)(O) \sim (P) + (d - 1)(O) \), then the divisor \( D = (P) - (O) \) is of degree 0 but is not the divisor of a rational function on \( E \) because \( P - O \neq O \),
• if \( m = 4 \), then \( H = \{ O, P_1, P_2, P_1 + P_2 \} \) and \( f^*(O) \sim (O) + (P_1) + (P_2) + (P_1 + P_2) + (d - 4)(O) \sim d(O) \) because \( \text{div}(y) = (P_1) + (P_2) + (P_1 + P_2) - 3(O) \).

\[ \square \]

**Remark 3.4.** Let \( E \) be the elliptic curve with affine model \( y^2 = x^3 + x \). It has complex multiplication by \([i] : (x, y) \to (-x, iy)\). Then the isogeny \([1+i] \) has degree 2, and as \([2] = [1+i][1-i] \), we get \( \text{Ker}[1+i] \subset E[2] \), hence \( \text{Card}(E[2] \cap \text{Ker}[1+i]) = 2 \), hence by Proposition 5.3 the morphism \([1+i] \) is not polarized. But as shown in [8], the morphism \([2+i] \) is polarized, another proof of this fact is that \([5] = [2+i][2-i] \), hence \( \text{Ker}[2+i] \subset E[5] \), but \( E[5] \cap E[2] = O \), hence again by Proposition 5.3 the morphism \([2+i] \) is polarized, with weight 5.

**Remark 3.5.** This result sharpens the remark made in the introduction of [9] concerning elliptic curves. See the remark [23] below for more details.

### 3.3. Explicit Lattès dynamical system.

Let \( E \) be the elliptic curve with affine model \( y^2 = x^3 + x \). It has complex multiplication by \([i] : (x, y) \to (-x, iy)\). Let \( \pi : E \to \mathbb{P}^1 \) be defined by \( \pi(x,y) = x \) and \( \pi(O) = \infty \). Then we get the following picture:

\[
E \xrightarrow{[2+i]} E \\
\pi \downarrow \quad \downarrow \pi \\
\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1
\]

A direct calculation shows that the Lattès map \( \varphi \) can be expressed on the affine chart as

\[
\varphi(x) = \frac{(3 - 4i)x(x^2 + 1 - 2i)^2}{(5x^2 + 1 + 2i)^2}.
\]

**Remark 3.6.** We thus get the \( x \)-coordinate of the four non-trivial \([2+i] \)-torsion points: \( \pm x = \sqrt{\frac{\sqrt{5} - 1}{10}} + i \frac{\sqrt{10}}{\sqrt{5} - 1} \).

For the Lattès map of \([2-i] \), we get on the affine chart

\[
\psi(x) = \frac{(3 + 4i)x(x^2 + 1 + 2i)^2}{(5x^2 + 1 - 2i)^2}.
\]

Now take a look at

\[
\delta : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\
(s,t) \longmapsto (\varphi(s), \psi(t)).
\]

Then if \( D = \{ \infty \} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{ \infty \} \), we get \( \delta^* D \sim 5D \). Hence \( \delta \) is polarized by \( D \) with weight 5. This gives an explicit counterexample to Conjecture [14] in the case of \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) as in the constructions of [4]. See [3] for Lattès dynamical systems.
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